

A PRIORI ESTIMATES FOR THE FREE BOUNDARY PROBLEM OF INCOMPRESSIBLE NEO-HOOKEAN ELASTODYNAMICS

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ABSTRACT. A free boundary problem for the incompressible neo-Hookean elastodynamics is studied in two and three spatial dimensions. The *a priori* estimates in Sobolev norms of solutions with the physical vacuum condition are established through a geometrical point of view of [3]. Some estimates on the second fundamental form and velocity of the free surface are also obtained.

1. INTRODUCTION

We are concerned with the motion of neo-Hookean elastic waves in an incompressible material for which the deformation or strain is proportional to the stress. Precisely, we consider the free boundary problem of the following incompressible elastodynamic equations of neo-Hookean elastic materials:

$$v_t + v \cdot \partial v + \partial p = \operatorname{div} (FF^\top), \quad (1.1a)$$

$$F_t + v \cdot \partial F = \partial v F, \quad (1.1b)$$

$$\operatorname{div} v = 0, \quad \operatorname{div} F^\top = 0, \quad (1.1c)$$

in a set

$$\mathcal{D} = \bigcup_{0 \leq t \leq T} \{t\} \times \mathcal{D}_t,$$

where $\mathcal{D}_t \subset \mathbb{R}^n, n = 2$ or 3 , is the domain that the material occupies at time $t \in [0, T]$ for some $T > 0$; where $\partial = (\partial_1, \dots, \partial_n)$ and div are the usual gradient operator and spatial divergence in the Eulerian coordinates with $\partial_i = \partial/\partial x^i$, respectively; $v(t, x) = (v_1(t, x), \dots, v_n(t, x))$ is the velocity vector field of the fluid, $p(t, x)$ is the pressure, $F(t, x) = (F_{ij}(t, x))$ is the deformation tensor, $F^\top = (F_{ji})$ denotes the transpose of the $n \times n$ matrix F , FF^\top is the Cauchy-Green tensor in the case of neo-Hookean elastic materials (cf. [10, 16]); and the notations $(\partial v)_{ij} = \partial_j v_i$, $(\partial v F)_{ij} = (\partial v F)^{ij} = (\partial v)_{ik} F^{kj} = \partial_k v^i F^{kj}$, $\operatorname{div} v = \partial_i v^i$, $(\operatorname{div} F^\top)^i = \partial_j F^{ji}$ follow the Einstein summation convention: $v^i = \delta^{ij} v_j = v_i$ and $F^{ij} = \delta^{ik} \delta^{jl} F_{kl} = F_{ij}$. The boundary conditions on the free boundary:

$$\partial \mathcal{D} = \bigcup_{0 \leq t \leq T} \{t\} \times \partial \mathcal{D}_t$$

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are prescribed as the following:

$$p=0 \quad \text{on } \partial\mathcal{D}, \quad (1.2a)$$

$$\mathcal{N} \cdot F^\top = 0 \quad \text{on } \partial\mathcal{D}, \quad (1.2b)$$

$$(\partial_t + v \cdot \partial)|_{\partial\mathcal{D}} \in T(\partial\mathcal{D}), \quad (1.2c)$$

where $\mathcal{N}(t, x)$ is the exterior unit normal to the free surface $\partial\mathcal{D}_t$ and $T(\partial\mathcal{D})$ is the tangential space to $\partial\mathcal{D}$. The boundary condition (1.2a) implies that the pressure p vanishes outside the domain, (1.2b) indicates that the normal component of F^\top (i.e., $\mathcal{N}_k F^{kj}$) vanishes on the boundary, and (1.2c) means that the free boundary moves with the velocity v of the material particles, i.e., $v \cdot \mathcal{N} = \kappa$ on $\partial\mathcal{D}_t$ with κ the normal velocity of $\partial\mathcal{D}_t$.

For a simply connected bounded domain $\mathcal{D}_0 \subset \mathbb{R}^n$ that is homeomorphic to the unit ball, and the initial data $(v_0(x), F_0(x))$ satisfying the constraint (1.1c): $\operatorname{div} v_0 = 0$, $\operatorname{div} F_0^\top = 0$, we shall establish *a priori* estimates for the set $\mathcal{D} \subset [0, T] \times \mathbb{R}^n$ and the vector fields v and F solving (1.1)-(1.2) with the initial conditions:

$$\{x : (0, x) \in \mathcal{D}\} = \mathcal{D}_0, \quad (v, F)|_{t=0} = (v_0(x), F_0(x)) \text{ for } x \in \mathcal{D}_0. \quad (1.3)$$

We will study the free boundary problem (1.1)-(1.3) under the following natural condition (cf. [2–4, 6, 9, 11–14, 17–19]):

$$\nabla_{\mathcal{N}} p \leq -\varepsilon < 0 \text{ on } \partial\mathcal{D}_t, \quad (1.4)$$

where $\nabla_{\mathcal{N}} = \mathcal{N}^i \partial_i$ and $\varepsilon > 0$ is a constant. We assume that (1.4) holds initially, and will verify that it still holds within a time period. Roughly speaking, the elastic body will not break up in the interior since the pressure is positive, the boundary moves according to the velocity, and the boundary is the level set of the pressure that, together with the Cauchy-Green tensor, determines the acceleration, thus the regularity of the boundary is quite involved, which is a difficult issue for this problem.

There have been some results for the free surface problem of the incompressible Euler equations of fluids in the recent decades, see for examples [1, 3, 4, 6, 11–14, 17–19] and the references therein. For elastodynamics, there have been some studies on the fixed boundary problems, see for examples Ebin [7, 8] for the global existence of small solutions to the three-dimensional incompressible and isotropic elasticity equations and the special case of incompressible neo-Hookean materials, and Sideris-Thomases [15, 16] for the global existence of the three-dimensional incompressible elasticity. In this paper, we shall prove the *a priori* estimates for the free boundary problem (1.1)-(1.3) in all physical spatial dimensions $n = 2, 3$ by adopting a geometrical point of view used in Christodoulou-Lindblad [3] and establishing estimates on quantities such as the second fundamental form and the velocity of the free surface.

Define the material derivative by $D_t = \partial_t + v^k \partial_k$. We rewrite the system (1.1) as

$$D_t v_i + \partial_i p = \partial_k F_{ij} F^{kj}, \quad \text{in } \mathcal{D}, \quad (1.5a)$$

$$D_t F_{ij} = \delta^{kl} \partial_k v_i F_{lj}, \quad \text{in } \mathcal{D}, \quad (1.5b)$$

$$\partial_i v^i = 0, \quad \partial_j F^{ji} = 0, \quad \text{in } \mathcal{D}. \quad (1.5c)$$

From (1.5), one has

$$\frac{1}{2} \frac{d}{dt} \int_{\mathcal{D}_t} (|v|^2 + |F|^2) dx = - \int_{\partial \mathcal{D}_t} p v^i \mathcal{N}_i dS + \int_{\partial \mathcal{D}_t} F_{ij} F^{kj} v^i \mathcal{N}_k dS, \quad (1.6)$$

where dS is the surface measure. We see that (1.6) and the boundary conditions (1.2) yield the conserved physical energy:

$$E_0(t) = \int_{\mathcal{D}_t} \left(\frac{1}{2} |v(t, x)|^2 + \frac{1}{2} |F(t, x)|^2 \right) dx. \quad (1.7)$$

Note that the identities $\operatorname{div} F^\top = 0$ in \mathcal{D} and $\mathcal{N} \cdot F^\top = 0$ on $\partial \mathcal{D}$ are preserved, that is, they hold if $\operatorname{div} F_0^\top = 0$ in \mathcal{D}_0 and $\mathcal{N} \cdot F_0^\top = 0$ on $\partial \mathcal{D}_0$ for initial data, where \mathcal{N} denotes the exterior unit normal to the initial interface $\partial \mathcal{D}_0$, which will be verified later in the Lagrangian coordinates.

The higher order energy norm has a boundary part and an interior part. Following the definitions and notations of [3], we define the boundary part through the orthogonal projection to the tangent space of the boundary. The orthogonal projection Π to the tangent space of the boundary of a $(0, r)$ tensor α is defined as the projection of each component in the normal direction, that is,

$$(\Pi \alpha)_{i_1 \dots i_r} = \Pi_{i_1}^{j_1} \dots \Pi_{i_r}^{j_r} \alpha_{j_1 \dots j_r}, \quad (1.8)$$

where $\Pi_i^j = \delta_i^j - \mathcal{N}_i \mathcal{N}^j$ with the convention $\mathcal{N}^j = \delta^{ij} \mathcal{N}_i = \mathcal{N}_j$.

Denote by $\bar{\partial}_i = \Pi_i^j \partial_j$ a tangential derivative. Then we see that for $q = 0$ on $\partial \mathcal{D}_t$, one has $\bar{\partial}_i q = 0$ on $\partial \mathcal{D}_t$ and

$$(\Pi \partial^2 q)_{ij} = \theta_{ij} \nabla \mathcal{N} q, \quad (1.9)$$

where $\theta_{ij} = \bar{\partial}_i \mathcal{N}_j$ is the second fundamental form of $\partial \mathcal{D}_t$.

Consider the following positive definite quadratic form Q of the form (see [3]):

$$Q(\alpha, \beta) = q^{i_1 j_1} \dots q^{i_r j_r} \alpha_{i_1 \dots i_r} \beta_{j_1 \dots j_r}, \quad (1.10)$$

where

$$q^{ij} = \delta^{ij} - \eta(d)^2 \mathcal{N}^i \mathcal{N}^j, \quad d(x) = \operatorname{dist}(x, \partial \mathcal{D}_t), \quad \text{and } \mathcal{N}^i = -\delta^{ij} \partial_j d, \quad (1.11)$$

and η is a smooth cutoff function satisfying

$$0 \leq \eta(d) \leq 1, \quad \eta(d) = 1 \text{ for } d < \frac{d_0}{4}, \quad \eta(d) = 0 \text{ for } d > \frac{d_0}{2},$$

with d_0 a fixed number less than the injectivity radius of the normal exponential map ι_0 which is the largest number ι_0 such that the map

$$\partial \mathcal{D}_t \times (-\iota_0, \iota_0) \rightarrow \{x \in \mathbb{R}^n : \operatorname{dist}(x, \partial \mathcal{D}_t) < \iota_0\}, \quad (1.12)$$

defined by

$$(\bar{x}, \iota) \rightarrow x = \bar{x} + \iota \mathcal{N}(\bar{x}),$$

is an injection. The quadratic form Q is the inner product of the tangential components when restricted to the boundary: $Q(\alpha, \beta) = \langle \Pi \alpha, \Pi \beta \rangle$, and $Q(\alpha, \alpha) = |\alpha|^2$ in the interior.

Let $\operatorname{sgn}(s)$ be the sign function of the real number s . Denote

$$(\operatorname{curl} F^\top)_{ijk} := \partial_i F_{jk} - \partial_j F_{ik}$$

and $\vartheta = (-\nabla_{\mathcal{N}} p)^{-1}$. Then, we define the higher order energies for $r \geq 1$ as:

$$\begin{aligned} E_r(t) &= \int_{\mathcal{D}_t} \left(\delta^{ij} Q(\partial^r v_i, \partial^r v_j) + \delta^{ij} \delta^{km} Q(\partial^r F_{ik}, \partial^r F_{jm}) \right) dx \\ &\quad + \int_{\mathcal{D}_t} \left(|\partial^{r-1} \operatorname{curl} v|^2 + |\partial^{r-1} \operatorname{curl} F^\top|^2 \right) dx \\ &\quad + \operatorname{sgn}(r-1) \int_{\partial \mathcal{D}_t} Q(\partial^r p, \partial^r p) \vartheta dS. \end{aligned} \quad (1.13)$$

The higher order energy norm has a boundary part (for $r \geq 2$) which controls the norms of the second fundamental form of the free surface, and an interior part which controls the norms of the velocity and thus the pressure. We will prove that the time derivatives of the energy norms are controlled by themselves. One advantage of the above higher order energy norms is that the time derivatives of the interior parts yield some boundary terms which have some cancellation with the leading-order terms in the time derivatives of the boundary integrals.

Now, we can state the main result of this paper as follows.

Theorem 1.1. *Let*

$$\mathcal{K}(0) = \max \left(\|\theta(0, \cdot)\|_{L^\infty(\partial \mathcal{D}_0)}, \frac{1}{\iota_0(0)} \right)$$

and

$$\mathcal{E}(0) = \left\| \frac{1}{\nabla_{\mathcal{N}} p(0, \cdot)} \right\|_{L^\infty(\partial \mathcal{D}_0)}.$$

Then, there exists a continuous function $\mathcal{T} > 0$ such that if

$$T \leq \mathcal{T}(\mathcal{K}(0), \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \operatorname{Vol} \mathcal{D}_0),$$

any smooth solution of the free boundary problem (1.1)-(1.4) satisfies

$$\sum_{s=0}^{n+1} E_s(t) \leq 2 \sum_{s=0}^{n+1} E_s(0), \quad 0 \leq t \leq T.$$

We remark that Theorem 1.1 extends the result of [3] for the Euler equations of incompressible flow to the elastodynamics (1.1). Our proof will be based on the geometric point of view following [3]. We need to develop new ingredients in the proof to handle the deformation F and the interaction with the velocity v , which requires some new thoughts. For the well-posedness of incompressible Euler equations we refer the readers to [11, 12] and the references therein. The well-posedness of the elastodynamics (1.1) is much harder. In this paper we shall explore all the symmetries of the equations and then we will be able to establish the sharp *a priori* estimates. Although the well-posedness does not follow directly, these estimates are crucial for the local existence of smooth solution for the system (1.1) which could be possibly obtained by improving the estimates of this paper together with the Nash-Moser technique.

The rest of the paper is organized as follows. In Section 2, we reformulate the problem to a fixed initial-boundary value problem in the Lagrangian coordinates. The Lagrangian transformation induces a Riemannian metric on \mathcal{D}_0 , for which we recall the time evolution properties derived in [3] and prove some new identities which will be used later. Section 3 is devoted to the first order energy estimates. In

Section 4, we derive the higher order energy estimates using the identities derived in Section 2. We justify the *a priori* assumptions in Section 5. For the sake of completeness and the convenience of the readers we add an appendix to state some estimates which are used in this paper but were already proved in [3].

2. REFORMULATION IN LAGRANGIAN COORDINATES

In this section, we shall introduce the Lagrangian coordinates to reformulate the free boundary problem to fix the boundaries.

Following the same terminology and lines of [3], we present the transformation between the Eulerian coordinates (t, x) and the Lagrangian coordinates (t, y) . For a velocity vector field $v(t, x)$ in $\mathcal{D} \subset [0, T] \times \mathbb{R}^n$ with $(1, v) \in T(\partial\mathcal{D})$ (i.e., the boundary $\partial\mathcal{D}_t$ moves with the velocity v), define $x = x(t, y) = f_t(y)$ as the trajectory of particles:

$$\begin{cases} \frac{dx}{dt} = v(t, x(t, y)), & (t, y) \in [0, T] \times \Omega, \\ x(0, y) = f_0(y), & y \in \Omega. \end{cases} \quad (2.1)$$

where, $f_0 : \Omega \rightarrow \mathcal{D}_0$ is some given diffeomorphism defined on a given simple domain Ω , and $f_t : \Omega \rightarrow \mathcal{D}_t$ with $f_t(y) = x(t, y)$ is a change of coordinates for each t . As a result, for each t from the Euclidean metric δ_{ij} in \mathcal{D}_t , a metric g_{ab} (with inverse g^{cd}) in Ω is induced as

$$g_{ab}(t, y) = \delta_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b}, \quad (2.2)$$

and thus the covariant differentiation ∇_a in the y_a -coordinates, $a = 0, \dots, n$, in Ω will be used for the metric $g_{ab}(t, y)$. An $(0, r)$ tensor $w(t, x)$ in the x -coordinates can be expressed as $k(t, y)$ in the y -coordinates:

$$k_{a_1 \dots a_r}(t, y) = \frac{\partial x^{i_1}}{\partial y^{a_1}} \dots \frac{\partial x^{i_r}}{\partial y^{a_r}} w_{i_1 \dots i_r}(t, x), \quad x = x(t, y),$$

and the covariant differentiation of the tensor $k(t, y)$ is the $(0, r+1)$ tensor:

$$\nabla_a k_{a_1 \dots a_r} = \frac{\partial x^i}{\partial y^a} \frac{\partial x^{i_1}}{\partial y^{a_1}} \dots \frac{\partial x^{i_r}}{\partial y^{a_r}} \frac{\partial w_{i_1 \dots i_r}}{\partial x^i} \quad (2.3)$$

with the invariant norms of tensors:

$$g^{a_1 b_1} \dots g^{a_r b_r} k_{a_1 \dots a_r} k_{b_1 \dots b_r} = \delta^{i_1 j_1} \dots \delta^{i_r j_r} w_{i_1 \dots i_r} w_{j_1 \dots j_r}. \quad (2.4)$$

From

$$\partial_i = \frac{\partial}{\partial x^i} = \frac{\partial y^a}{\partial x^i} \frac{\partial}{\partial y^a}, \quad (2.5)$$

the curvature vanishes in both the x -coordinates and the y -coordinates, and then $[\nabla_a, \nabla_b] = 0$. Denote

$$k_{a \dots}{}^b{}_{\dots c} = g^{bd} k_{a \dots d \dots c}$$

and

$$D_t = \frac{\partial}{\partial t} \Big|_{y=\text{const}} = \frac{\partial}{\partial t} \Big|_{x=\text{const}} + v^k \frac{\partial}{\partial x^k}.$$

Since covariant differentiation commutes with lowering and rising indices:

$$g^{ce}\nabla_a k_{b\cdots d} = \nabla_a g^{ce} k_{b\cdots d},$$

one has, from [3, Lemma 2.2],

$$D_t k_{a_1\cdots a_r} = \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \left(D_t w_{i_1\cdots i_r} + \frac{\partial v^\ell}{\partial x^{i_1}} w_{\ell\cdots i_r} + \cdots + \frac{\partial v^\ell}{\partial x^{i_r}} w_{i_1\cdots \ell} \right). \quad (2.6)$$

We now recall a result of [3] concerning time derivatives of the change of coordinates and commutators between time derivatives and space derivatives:

Lemma 2.1 ([3, Lemma 2.1]). *Let $x = f_t(y)$ be the change of variables given by (2.1), and let g_{ab} be the metric given by (2.2). Let $v_i = \delta_{ij} v^j = v^i$, and set*

$$u_a(t, y) = v_i(t, x) \frac{\partial x^i}{\partial y^a}, \quad u^a = g^{ab} u_b, \quad h_{ab} = \frac{1}{2} D_t g_{ab}, \quad h^{ab} = g^{ac} h_{cd} g^{db}. \quad (2.7)$$

Then

$$D_t \frac{\partial x^i}{\partial y^a} = \frac{\partial x^k}{\partial y^a} \frac{\partial v^i}{\partial x^k}, \quad D_t \frac{\partial y^a}{\partial x^i} = -\frac{\partial y^a}{\partial x^k} \frac{\partial v^k}{\partial x^i}, \quad (2.8)$$

and

$$D_t g_{ab} = \nabla_a u_b + \nabla_b u_a, \quad D_t g^{ab} = -2h^{ab}, \quad D_t d\mu_g = g^{ab} h_{ab} d\mu_g, \quad (2.9)$$

where $d\mu_g$ is the Riemannian volume element on Ω in the metric g .

We also recall from [3] the estimates of commutators between the material derivative D_t and space derivatives ∂_i and covariant derivatives ∇_a :

Lemma 2.2 ([3, Lemma 2.3]). *Let ∂_i be given by (2.5). Then*

$$[D_t, \partial_i] = -(\partial_i v^k) \partial_k. \quad (2.10)$$

Furthermore,

$$[D_t, \partial^r] = \sum_{s=0}^{r-1} -\binom{r}{s+1} (\partial^{1+s} v) \cdot \partial^{r-s}, \quad (2.11)$$

where the symmetric dot product is defined to be in components

$$((\partial^{1+s} v) \cdot \partial^{r-s})_{i_1\cdots i_r} = \frac{1}{r!} \sum_{\sigma \in \Sigma_r} \left(\partial_{i_{\sigma_1}\cdots i_{\sigma_{1+s}}}^{1+s} v^k \right) \partial_{k i_{\sigma_{s+2}}\cdots i_{\sigma_r}}^{r-s}, \quad (2.12)$$

and Σ_r denotes the collection of all permutations of $\{1, 2, \dots, r\}$.

Lemma 2.3. *Let $T_{a_1\cdots a_r}$ be a $(0, r)$ tensor. Then*

$$[D_t, \nabla_a] T_{a_1\cdots a_r} = -(\nabla_{a_1} \nabla_a u^d) T_{da_2\cdots a_r} - \cdots - (\nabla_{a_r} \nabla_a u^d) T_{a_1\cdots a_{r-1}d}. \quad (2.13)$$

In particular,

$$[D_t, g^{ab} \nabla_a] T_b = -2h^{ab} \nabla_a T_b - (\Delta u^e) T_e, \quad (2.14)$$

$$\begin{aligned} [D_t, g^{ac} g^{bd} \nabla_a] T_{cd} = & -\Delta u_e T^{eb} - g^{bd} \nabla_d \nabla_a u_e T^{ae} - g^{ae} \nabla_e u_f \nabla_a T^{fb} \\ & - g^{be} \nabla_e u_f \nabla_a T^{af} - \nabla_f u^a \nabla_a T^{fb} - \nabla_f u^b \nabla_a T^{af}. \end{aligned} \quad (2.15)$$

If ∇ denotes the first order covariant derivative, $\Delta = g^{cd}\nabla_c\nabla_d$ is the Laplacian operator under the Lagrangian coordinates and q is a function, then

$$[D_t, \nabla]q = 0, \quad [D_t, \Delta]q = -2h^{ab}\nabla_a\nabla_b q - (\Delta u^e)\nabla_e q, \quad (2.16)$$

and

$$[D_t, \nabla^r]q = \sum_{s=1}^{r-1} - \binom{r}{s+1} (\nabla^{s+1}u) \cdot \nabla^{r-s}q, \quad (2.17)$$

where the symmetric dot product is defined as, in components,

$$((\nabla^{s+1}u) \cdot \nabla^{r-s}q)_{a_1 \dots a_r} = \frac{1}{r!} \sum_{\sigma \in \Sigma_r} \left(\nabla_{a_{\sigma_1} \dots a_{\sigma_{s+1}}}^{s+1} u^d \right) \nabla_{da_{\sigma_{s+2}} \dots a_{\sigma_r}}^{r-s} q. \quad (2.18)$$

Proof. From (2.9) and (2.13), it follows that

$$\begin{aligned} & [D_t, g^{ac}g^{bd}\nabla_a]T_{cd} \\ &= D_t g^{ac}g^{bd}\nabla_a T_{cd} + g^{ac}D_t g^{bd}\nabla_a T_{cd} + g^{ac}g^{bd}D_t \nabla_a T_{cd} - g^{ac}g^{bd}\nabla_a D_t T_{cd} \\ &= -2h^{ac}g^{bd}\nabla_a T_{cd} - 2g^{ac}h^{bd}\nabla_a T_{cd} - g^{ac}g^{bd}(\nabla_c \nabla_a u^e T_{ed} + \nabla_d \nabla_a u^e T_{ce}) \\ &= -\Delta u_e T^{eb} - g^{bd}\nabla_d \nabla_a u_e T^{ae} - g^{ae}\nabla_e u_f \nabla_a T^{fb} \\ &\quad - g^{be}\nabla_e u_f \nabla_a T^{af} - \nabla_f u^a \nabla_a T^{fb} - \nabla_f u^b \nabla_a T^{af}. \end{aligned}$$

For other identities, one can see the proofs in [3, Lemma 2.4] and [9, Lemma 2.3], and we omit the details. \square

Let

$$\mathbb{F}_{ab} = F_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b}, \quad \mathbb{F}^{ab} = g^{ac}g^{bd}\mathbb{F}_{cd}, \quad |\mathbb{F}|^2 = \mathbb{F}_{ab}\mathbb{F}^{ab}.$$

Then, it follows from (2.4) that

$$|\mathbb{F}|^2 = |F|^2 \quad \text{and} \quad F_{ij} = \frac{\partial y^a}{\partial x^i} \frac{\partial y^b}{\partial x^j} \mathbb{F}_{ab}. \quad (2.19)$$

From (2.7), (2.19), (2.8) and (2.3), it follows that

$$\begin{aligned} D_t u_a &= D_t \left(v_i \frac{\partial x^i}{\partial y^a} \right) = \frac{\partial x^i}{\partial y^a} D_t v_i + v_i D_t \frac{\partial x^i}{\partial y^a} \\ &= -\nabla_a p + \frac{\partial y^b}{\partial x^l} \frac{\partial y^c}{\partial x^k} \nabla_c \mathbb{F}_{ab} \delta^{jk} \frac{\partial y^e}{\partial x^j} \delta^{ml} \frac{\partial y^f}{\partial x^m} \mathbb{F}_{ef} + u_b \delta^{ij} \frac{\partial y^b}{\partial x^j} \frac{\partial y^c}{\partial x^i} \nabla_a u_c \\ &= -\nabla_a p + \nabla_c \mathbb{F}_{ab} \mathbb{F}^{cb} + u^c \nabla_a u_c. \end{aligned}$$

Similarly,

$$\begin{aligned} D_t \mathbb{F}_{ab} &= D_t \left(F_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} \right) = \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} D_t F_{ij} + F_{ij} \left(D_t \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} + \frac{\partial x^i}{\partial y^a} D_t \frac{\partial x^j}{\partial y^b} \right) \\ &= \nabla_d u_a \frac{\partial y^d}{\partial x^k} \frac{\partial y^c}{\partial x^l} \delta^{kl} \mathbb{F}_{cb} + \mathbb{F}_{cb} \frac{\partial y^c}{\partial x^i} \nabla_a u_e \frac{\partial y^e}{\partial x^j} \delta^{ij} + \mathbb{F}_{ac} \frac{\partial y^c}{\partial x^j} \nabla_b u_d \frac{\partial y^d}{\partial x^i} \delta^{ij} \\ &= g^{dc} \nabla_d u_a \mathbb{F}_{cb} + \mathbb{F}_{cb} \nabla_a u^c + \mathbb{F}_{ac} \nabla_b u^c. \end{aligned}$$

Then, we can rewrite the system (1.1) in the Lagrangian coordinates as

$$D_t u_a + \nabla_a p = \nabla_c \mathbb{F}_{ab} \mathbb{F}^{cb} + u^c \nabla_a u_c \quad \text{in } [0, T] \times \Omega, \quad (2.20a)$$

$$D_t \mathbb{F}_{ab} = g^{dc} \nabla_d u_a \mathbb{F}_{cb} + \mathbb{F}_{cb} \nabla_a u^c + \mathbb{F}_{ac} \nabla_b u^c \quad \text{in } [0, T] \times \Omega, \quad (2.20b)$$

$$\nabla_a u^a = 0, \quad \nabla_a \mathbb{F}^{ab} = 0 \quad \text{in } [0, T] \times \Omega, \quad (2.20c)$$

$$p = 0, \quad N^a \mathbb{F}_{ab} = 0 \quad \text{on } [0, T] \times \partial\Omega. \quad (2.20d)$$

From (1.7), we also have the conserved energy

$$E_0(t) = \int_{\Omega} \left(\frac{1}{2} |u(t, y)|^2 + \frac{1}{2} |\mathbb{F}(t, y)|^2 \right) d\mu_g. \quad (2.21)$$

We note that if

$$|\nabla u(t, y)| \leq C \quad \text{in } [0, T] \times \overline{\Omega}, \quad (2.22)$$

and $\operatorname{div} v = 0$ in $[0, T] \times \Omega$, then the divergence free property of \mathbb{F}^\top , i.e., $\operatorname{div} \mathbb{F}^\top = 0$, is preserved for all times under the Lagrangian coordinates or in view of the material derivative, i.e., $D_t \operatorname{div} \mathbb{F}^\top = 0$. Indeed, from (2.15) and Lemma 2.1, the divergence of (2.20b) gives

$$\begin{aligned} D_t \nabla_a \mathbb{F}^{ab} &= D_t (g^{ac} g^{bd} \nabla_a \mathbb{F}_{cd}) = [D_t, g^{ac} g^{bd} \nabla_a] \mathbb{F}_{cd} + g^{ac} g^{bd} \nabla_a D_t \mathbb{F}_{cd} \\ &= -\Delta u_e \mathbb{F}^{eb} - g^{bd} \nabla_d \nabla_a u_e \mathbb{F}^{ae} - g^{ae} \nabla_e u_f \nabla_a \mathbb{F}^{fb} - g^{be} \nabla_e u_f \nabla_a \mathbb{F}^{af} \\ &\quad - \nabla_f u^a \nabla_a \mathbb{F}^{fb} - \nabla_f u^b \nabla_a \mathbb{F}^{af} + \nabla_f \nabla_a u^a \mathbb{F}^{fb} + \nabla_f u^a \nabla_a \mathbb{F}^{fb} \\ &\quad + g^{ac} \nabla_a \mathbb{F}^{eb} \nabla_c u_e + \mathbb{F}^{eb} \Delta u_e + g^{bd} \nabla_a \mathbb{F}^{ae} \nabla_d u_e + g^{bd} \mathbb{F}^{ae} \nabla_a \nabla_d u_e \\ &= -\nabla_f u^b \nabla_a \mathbb{F}^{af}, \end{aligned}$$

which implies, by the Gronwall inequality and the identity $|D_t f| = |D_t f|$, that

$$|\nabla_a \mathbb{F}^{ab}(t, y)| \leq e^{Ct} |\nabla_a \mathbb{F}^{ab}(0, y)| = 0.$$

Moreover, $N \cdot \mathbb{F}^\top = 0$ is also preserved for all time t in the lifespan $[0, T]$, that is, we have $N^a \mathbb{F}_{ab} = 0$ on $[0, T] \times \partial\Omega$ if $N \cdot \mathbb{F}^\top = 0$ on the boundary $\partial\Omega$ at initial time. In fact, we have, from (2.20b), Lemmas 2.1 and A.4, that

$$\begin{aligned} D_t (N^a \mathbb{F}_{ab}) &= D_t (g^{ac} N_c \mathbb{F}_{ab}) = D_t g^{ac} N_c \mathbb{F}_{ab} + g^{ac} D_t N_c \mathbb{F}_{ab} + N^a D_t \mathbb{F}_{ab} \\ &= -2h^{ac} N_c \mathbb{F}_{ab} + g^{ac} h_{NN} N_c \mathbb{F}_{ab} + N^a (g^{dc} \nabla_d u_a \mathbb{F}_{cb} + \mathbb{F}_{cb} \nabla_a u^c + \mathbb{F}_{ac} \nabla_b u^c) \\ &= -g^{ad} \nabla_d u^c N_c \mathbb{F}_{ab} - \nabla_e u^a N^e \mathbb{F}_{ab} + h_{NN} N^a \mathbb{F}_{ab} + g^{dc} \nabla_d u_a N^a \mathbb{F}_{cb} \\ &\quad + \mathbb{F}_{cb} \nabla_a u^c N^a + \mathbb{F}_{ac} \nabla_b u^c N^a \\ &= h_{NN} N^a \mathbb{F}_{ab} + \nabla_b u^c N^a \mathbb{F}_{ac}, \end{aligned}$$

which yields similarly that

$$|(N^a \mathbb{F}_{ab})(t, y)| \leq e^{Ct} |(N^a \mathbb{F}_{ab})(0, y)| = 0.$$

3. THE FIRST ORDER ENERGY ESTIMATES

In this section, we prove the first order energy estimate. From (2.13) and (2.20a), we get

$$\begin{aligned} D_t (\nabla_b u_a) + \nabla_b \nabla_a p &= -(\nabla_a \nabla_b u^d) u_d + \nabla_b (\nabla_c \mathbb{F}_{ad} \mathbb{F}^{cd}) + \nabla_b (u^c \nabla_a u_c) \\ &= \nabla_b u^c \nabla_a u_c + \nabla_b \nabla_c \mathbb{F}_{ad} \mathbb{F}^{cd} + \nabla_c \mathbb{F}_{ad} \nabla_b \mathbb{F}^{cd}. \end{aligned}$$

From (2.13) and (2.20b), we obtain

$$\begin{aligned} D_t(\nabla_c \mathbb{F}_{ab}) &= -(\nabla_a \nabla_c u^d) \mathbb{F}_{db} - (\nabla_b \nabla_c u^d) \mathbb{F}_{ad} + g^{de} \nabla_c \nabla_d u_a \mathbb{F}_{eb} + g^{de} \nabla_d u_a \nabla_c \mathbb{F}_{eb} \\ &\quad + \nabla_c \mathbb{F}_{db} \nabla_a u^d + \mathbb{F}_{db} \nabla_c \nabla_a u^d + \nabla_c \mathbb{F}_{ad} \nabla_b u^d + \mathbb{F}_{ad} \nabla_b \nabla_c u^d \\ &= g^{de} \nabla_c \mathbb{F}_{eb} (\nabla_d u_a + \nabla_a u_d) + \nabla_c \mathbb{F}_{ad} \nabla_b u^d + g^{de} \mathbb{F}_{eb} \nabla_c \nabla_d u_a. \end{aligned}$$

Thus, we have

$$D_t(\nabla_b u_a) + \nabla_b \nabla_a p = \nabla_b u^c \nabla_a u_c + \nabla_b \nabla_c \mathbb{F}_{ad} \mathbb{F}^{cd} + \nabla_c \mathbb{F}_{ad} \nabla_b \mathbb{F}^{cd}, \quad (3.1)$$

and

$$D_t(\nabla_c \mathbb{F}_{ab}) = g^{de} \nabla_c \mathbb{F}_{eb} (\nabla_d u_a + \nabla_a u_d) + \nabla_c \mathbb{F}_{ad} \nabla_b u^d + g^{de} \mathbb{F}_{eb} \nabla_c \nabla_d u_a. \quad (3.2)$$

We now derive the material derivative of $g^{bd} \gamma^{ae} \nabla_a u_b \nabla_e u_d$. From (2.9), (2.7) and (A.9), we have

$$\begin{aligned} &D_t(g^{bd} \gamma^{ae} \nabla_a u_b \nabla_e u_d) \\ &= (D_t g^{bd}) \gamma^{ae} \nabla_a u_b \nabla_e u_d + g^{bd} (D_t \gamma^{ae}) \nabla_a u_b \nabla_e u_d + 2g^{bd} \gamma^{ae} (D_t \nabla_a u_b) \nabla_e u_d \\ &= -2\gamma^{ae} \gamma^{fc} \nabla_e u_f \nabla_a u^d \nabla_c u_d - 2\gamma^{ae} \nabla_e u^b \nabla_a \nabla_b p \\ &\quad + 2\gamma^{ae} \nabla_e u^b (\nabla_a \nabla_c \mathbb{F}_{bf} \mathbb{F}^{cf} + \nabla_c \mathbb{F}_{bf} \nabla_a \mathbb{F}^{cf}), \end{aligned} \quad (3.3)$$

and from (3.2) it follows that

$$\begin{aligned} &D_t(g^{bd} g^{cf} \gamma^{ae} \nabla_a \mathbb{F}_{bc} \nabla_e \mathbb{F}_{df}) \\ &= -2g^{bq} h_{qs} g^{sd} g^{cf} \gamma^{ae} \nabla_a \mathbb{F}_{bc} \nabla_e \mathbb{F}_{df} - 2g^{bd} g^{cq} h_{qs} g^{sf} \gamma^{ae} \nabla_a \mathbb{F}_{bc} \nabla_e \mathbb{F}_{df} \\ &\quad - 2g^{bd} g^{cf} \gamma^{aq} h_{qs} \gamma^{se} \nabla_a \mathbb{F}_{bc} \nabla_e \mathbb{F}_{df} + 2g^{bd} g^{cf} \gamma^{ae} D_t \nabla_a \mathbb{F}_{bc} \nabla_e \mathbb{F}_{df} \\ &= -2\nabla_q u_s \gamma^{aq} \gamma^{se} \nabla_a \mathbb{F}_{df} \nabla_e \mathbb{F}_{df} + 2\nabla_q u^d \gamma^{ae} \nabla_a \mathbb{F}^{qf} \nabla_e \mathbb{F}_{df} \\ &\quad + 2\gamma^{ae} \nabla_e \mathbb{F}_{bc} \mathbb{F}^{qc} \nabla_a \nabla_q u^b. \end{aligned} \quad (3.4)$$

Combining (3.3) with (3.4) yields

$$\begin{aligned} &D_t \left(g^{bd} \gamma^{ae} \nabla_a u_b \nabla_e u_d + g^{bd} g^{cf} \gamma^{ae} \nabla_a \mathbb{F}_{bc} \nabla_e \mathbb{F}_{df} \right) \\ &= -2\gamma^{ae} \gamma^{fc} \nabla_e u_f \nabla_a u^d \nabla_c u_d - 2\gamma^{ae} \gamma^{fc} \nabla_e u_f \nabla_a \mathbb{F}^{ds} \nabla_c \mathbb{F}_{ds} \\ &\quad - 2\nabla_b (\gamma^{ae} \nabla_e u^b \nabla_a p - \gamma^{ae} \nabla_e u^c \nabla_a \mathbb{F}_{cf} \mathbb{F}^{bf}) \\ &\quad + 2(\nabla_b \gamma^{ae}) (\nabla_e u^b \nabla_a p - \nabla_e u^c \nabla_a \mathbb{F}_{cf} \mathbb{F}^{bf}) \\ &\quad + 2\gamma^{ae} \nabla_e u^d \nabla_c \mathbb{F}_{df} \nabla_a \mathbb{F}^{cf} + 2\gamma^{ae} \nabla_c u^d \nabla_e \mathbb{F}_{df} \nabla_a \mathbb{F}^{cf}. \end{aligned} \quad (3.5)$$

Then we calculate the material derivatives of $|\operatorname{curl} u|^2$ and $|\operatorname{curl} \mathbb{F}^\top|^2$ where $\operatorname{curl} \mathbb{F}^\top$ is defined as

$$(\operatorname{curl} \mathbb{F}^\top)_{abc} := \nabla_a \mathbb{F}_{bc} - \nabla_b \mathbb{F}_{ac}.$$

Indeed, one has

$$\begin{aligned} D_t |\operatorname{curl} u|^2 &= D_t \left(g^{ac} g^{bd} (\operatorname{curl} u)_{ab} (\operatorname{curl} u)_{cd} \right) \\ &= 2(D_t g^{ac}) g^{bd} (\operatorname{curl} u)_{ab} (\operatorname{curl} u)_{cd} + 4g^{ac} g^{bd} (D_t \nabla_a u_b) (\operatorname{curl} u)_{cd} \\ &= -4g^{ae} g^{bd} \nabla_e u^c (\operatorname{curl} u)_{ab} (\operatorname{curl} u)_{cd} \end{aligned}$$

$$+ 4g^{ac}g^{bd}(\operatorname{curl} u)_{cd}(\nabla_a \mathbb{F}^{ef} \nabla_e \mathbb{F}_{bf} + \nabla_a \nabla_e \mathbb{F}_{bf} \mathbb{F}^{ef}),$$

and

$$\begin{aligned} D_t |\operatorname{curl} \mathbb{F}^\top|^2 &= D_t \left(g^{ac} g^{bd} g^{ef} (\operatorname{curl} \mathbb{F}^\top)_{abe} (\operatorname{curl} \mathbb{F}^\top)_{cdf} \right) \\ &= -4g^{aq} g^{bd} g^{ef} \nabla_q u^c (\operatorname{curl} \mathbb{F}^\top)_{abe} (\operatorname{curl} \mathbb{F}^\top)_{cdf} \\ &\quad - 2g^{ac} g^{bd} g^{eq} \nabla_q u^f (\operatorname{curl} \mathbb{F}^\top)_{abe} (\operatorname{curl} \mathbb{F}^\top)_{cdf} \\ &\quad + 4g^{ac} (\operatorname{curl} \mathbb{F}^\top)_{cdf} \nabla_a \mathbb{F}^{sf} \nabla_s u^d + 4g^{ac} g^{bd} (\operatorname{curl} \mathbb{F}^\top)_{cdf} \nabla_a \mathbb{F}^{qf} \nabla_b u_q \\ &\quad + 4g^{ac} g^{ef} (\operatorname{curl} \mathbb{F}^\top)_{cdf} \nabla_a \mathbb{F}^{ds} \nabla_e u_s + 4g^{ac} (\operatorname{curl} \mathbb{F}^\top)_{cdf} \mathbb{F}^{sf} \nabla_a \nabla_s u^d. \end{aligned}$$

Thus, we have obtained

$$\begin{aligned} &D_t (|\operatorname{curl} u|^2 + |\operatorname{curl} \mathbb{F}^\top|^2) \\ &= -4g^{ae} g^{bd} \nabla_e u^c (\operatorname{curl} u)_{ab} (\operatorname{curl} u)_{cd} + 4g^{ac} g^{bd} (\operatorname{curl} u)_{cd} \nabla_a \mathbb{F}^{ef} \nabla_e \mathbb{F}_{bf} \\ &\quad - 4g^{aq} g^{bd} g^{ef} \nabla_q u^c (\operatorname{curl} \mathbb{F}^\top)_{abe} (\operatorname{curl} \mathbb{F}^\top)_{cdf} \\ &\quad - 2g^{ac} g^{bd} g^{eq} \nabla_q u^f (\operatorname{curl} \mathbb{F}^\top)_{abe} (\operatorname{curl} \mathbb{F}^\top)_{cdf} \\ &\quad + 4g^{ac} (\operatorname{curl} \mathbb{F}^\top)_{cdf} \nabla_a \mathbb{F}^{sf} \nabla_s u^d + 4g^{ac} g^{bd} (\operatorname{curl} \mathbb{F}^\top)_{cdf} \nabla_a \mathbb{F}^{qf} \nabla_b u_q \\ &\quad + 4g^{ac} g^{ef} (\operatorname{curl} \mathbb{F}^\top)_{cdf} \nabla_a \mathbb{F}^{ds} \nabla_e u_s + 4 \nabla_e \left[g^{ac} g^{bd} (\operatorname{curl} u)_{cd} \nabla_a \mathbb{F}_{bf} \mathbb{F}^{ef} \right]. \end{aligned} \quad (3.6)$$

Define the first order energy as

$$\begin{aligned} E_1(t) &= \int_{\Omega} \left(g^{bd} \gamma^{ae} \nabla_a u_b \nabla_e u_d + g^{bd} g^{cf} \gamma^{ae} \nabla_a \mathbb{F}_{bc} \nabla_e \mathbb{F}_{df} \right) d\mu_g \\ &\quad + \int_{\Omega} \left(|\operatorname{curl} u|^2 + |\operatorname{curl} \mathbb{F}^\top|^2 \right) d\mu_g. \end{aligned} \quad (3.7)$$

Recall the Gauss formula for Ω and $\partial\Omega$:

$$\int_{\Omega} \nabla_a w^a d\mu_g = \int_{\partial\Omega} N_a w^a d\mu_\gamma, \quad \text{and} \quad \int_{\partial\Omega} \bar{\nabla}_a \bar{f}^a d\mu_\gamma = 0 \quad (3.8)$$

if \bar{f} is tangential to $\partial\Omega$ and N_a denotes the unit conormal to $\partial\Omega$. Then, we can establish the following estimate on the first order energy:

Theorem 3.1. *For any smooth solution of system (2.20) for $0 \leq t \leq T$ satisfying*

$$|\nabla p| \leq M, \quad |\nabla u| \leq M, \quad \text{in } [0, T] \times \Omega, \quad (3.9)$$

$$|\theta| + |\nabla u| + \frac{1}{\iota_0} \leq K, \quad \text{on } [0, T] \times \partial\Omega, \quad (3.10)$$

one has, for any $t \in [0, T]$,

$$E_1(t) \leq 2e^{CMt} E_1(0) + CK^2 (\operatorname{Vol} \Omega + E_0(0)) (e^{CMt} - 1) \quad (3.11)$$

with some constant $C > 0$ which depends only on the dimension n .

Proof. It follows, from (3.5), (3.6) and Gauss' formula, that

$$\begin{aligned} \frac{d}{dt} E_1(t) &= \int_{\Omega} D_t \left(g^{bd} \gamma^{ae} \nabla_a u_b \nabla_e u_d + g^{bd} g^{cf} \gamma^{ae} \nabla_a \mathbb{F}_{bc} \nabla_e \mathbb{F}_{df} \right) d\mu_g \\ &\quad + \int_{\Omega} \left(g^{bd} \gamma^{ae} \nabla_a u_b \nabla_e u_d + g^{bd} g^{cf} \gamma^{ae} \nabla_a \mathbb{F}_{bc} \nabla_e \mathbb{F}_{df} \right) \operatorname{tr} h d\mu_g \end{aligned}$$

$$\begin{aligned}
& + \int_{\Omega} D_t \left(|\operatorname{curl} u|^2 + |\operatorname{curl} \mathbb{F}^\top|^2 \right) d\mu_g \\
& + \int_{\Omega} \left(|\operatorname{curl} u|^2 + |\operatorname{curl} \mathbb{F}^\top|^2 \right) \operatorname{tr} h d\mu_g \\
= & - 2 \int_{\Omega} \gamma^{ae} \gamma^{fc} \nabla_e u_f \nabla_a u^d \nabla_c u_d d\mu_g - 2 \int_{\Omega} \gamma^{ae} \gamma^{fc} \nabla_e u_f \nabla_a \mathbb{F}^{ds} \nabla_c \mathbb{F}_{ds} d\mu_g \\
& - 2 \int_{\partial\Omega} N_b (\gamma^{ae} \nabla_e u^b \nabla_a p - \gamma^{ae} \nabla_e u^c \nabla_a \mathbb{F}_{cf} \mathbb{F}^{bf}) d\mu_\gamma
\end{aligned} \tag{3.12}$$

$$+ 2 \int_{\Omega} (\nabla_b \gamma^{ae}) (\nabla_e u^b \nabla_a p - \nabla_e u^c \nabla_a \mathbb{F}_{cf} \mathbb{F}^{bf}) d\mu_g \tag{3.13}$$

$$\begin{aligned}
& + 2 \int_{\Omega} \gamma^{ae} \nabla_e u^d \nabla_c \mathbb{F}_{df} \nabla_a \mathbb{F}^{cf} d\mu_g + 2 \int_{\Omega} \gamma^{ae} \nabla_e u^d \nabla_e \mathbb{F}_{df} \nabla_a \mathbb{F}^{cf} d\mu_g \\
& - 4 \int_{\Omega} g^{ae} g^{bd} \nabla_e u^c (\operatorname{curl} u)_{ab} (\operatorname{curl} u)_{cd} d\mu_g + 4 \int_{\Omega} g^{ac} g^{bd} (\operatorname{curl} u)_{cd} \nabla_a \mathbb{F}^{ef} \nabla_e \mathbb{F}_{bf} d\mu_g \\
& - 4 \int_{\Omega} g^{aq} g^{bd} g^{ef} \nabla_q u^c (\operatorname{curl} \mathbb{F}^\top)_{abe} (\operatorname{curl} \mathbb{F}^\top)_{cdf} d\mu_g \\
& - 2 \int_{\Omega} g^{ac} g^{bd} g^{eq} \nabla_q u^f (\operatorname{curl} \mathbb{F}^\top)_{abe} (\operatorname{curl} \mathbb{F}^\top)_{cdf} d\mu_g \\
& + 4 \int_{\Omega} g^{ac} (\operatorname{curl} \mathbb{F}^\top)_{cdf} \nabla_a \mathbb{F}^{sf} \nabla_s u^d d\mu_g + 4 \int_{\Omega} g^{ac} g^{bd} (\operatorname{curl} \mathbb{F}^\top)_{cdf} \nabla_a \mathbb{F}^{qf} \nabla_b u_q d\mu_g \\
& + 4 \int_{\Omega} g^{ac} g^{ef} (\operatorname{curl} \mathbb{F}^\top)_{cdf} \nabla_a \mathbb{F}^{ds} \nabla_e u_s d\mu_g \\
& + 4 \int_{\partial\Omega} N_e \mathbb{F}^{ef} g^{ac} g^{bd} (\operatorname{curl} u)_{cd} \nabla_a \mathbb{F}_{bf} d\mu_\gamma \\
& + \int_{\Omega} \left(g^{bd} \gamma^{ae} \nabla_a u_b \nabla_e u_d + g^{bd} g^{cf} \gamma^{ae} \nabla_a \mathbb{F}_{bc} \nabla_e \mathbb{F}_{df} \right) \operatorname{tr} h d\mu_g \\
& + \int_{\Omega} \left(|\operatorname{curl} u|^2 + |\operatorname{curl} \mathbb{F}^\top|^2 \right) \operatorname{tr} h d\mu_g.
\end{aligned} \tag{3.14}$$

Since $p = 0$ on the boundary $\partial\Omega$, it follows that $\overline{\nabla} p = 0$, i.e., $\gamma_a^d \nabla_d p = 0$, and then $\gamma^{ae} \nabla_a p = g^{ce} \gamma_c^a \nabla_a p = 0$ on $\partial\Omega$. In addition, $N \cdot \mathbb{F}^\top = 0$ on $\partial\Omega$. Thus, the integrals in (3.12) and (3.14) vanish.

For the term (3.13), we first have from (A.5) and (A.3),

$$\theta_{ab} = (\delta_a^c - N_a N^c) \nabla_c N_b = \nabla_a N_b - N_a \nabla_N N_b = \nabla_a N_b$$

since $\nabla_N N = 0$ in geodesic coordinates, and then

$$\begin{aligned}
\nabla_b \gamma^{ae} &= \nabla_b (g^{ae} - N^a N^e) = -\nabla_b (N^a N^e) \\
&= -(\nabla_b N^a) N^e - (\nabla_b N^e) N^a = -\theta_b^a N^e - \theta_b^e N^a.
\end{aligned}$$

Thus, by the Hölder inequality, (3.10) and Lemma A.5, one has

$$\begin{aligned}
& |(3.13)| \\
& \leq CK \left(\|\nabla u\|_{L^2(\Omega)} \|\nabla p\|_{L^\infty(\Omega)} (\operatorname{Vol} \Omega)^{1/2} + \|\nabla u\|_{L^\infty(\Omega)} \|\mathbb{F}\|_{L^2(\Omega)} \|\nabla \mathbb{F}\|_{L^2(\Omega)} \right) \\
& \leq CKM \left((\operatorname{Vol} \Omega)^{1/2} + E_0^{1/2}(0) \right) E_1^{1/2}(t).
\end{aligned}$$

For other terms, we can use the Hölder inequality directly. Hence, we obtain

$$\begin{aligned}
& \frac{d}{dt} E_1(t) \\
& \leq CKM \left((\text{Vol } \Omega)^{1/2} + E_0^{1/2}(0) \right) E_1^{1/2}(t) \\
& \quad + C \|\nabla u\|_{L^\infty(\Omega)} \left(\|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla \mathbb{F}\|_{L^2(\Omega)}^2 + \|\text{curl } u\|_{L^2(\Omega)}^2 + \left\| \text{curl } \mathbb{F}^\top \right\|_{L^2(\Omega)}^2 \right) \\
& \leq CKM \left((\text{Vol } \Omega)^{1/2} + E_0^{1/2}(0) \right) E_1^{1/2}(t) + CM E_1(t).
\end{aligned}$$

By the Gronwall inequality, we have

$$E_1^{1/2}(t) \leq e^{CMt/2} E_1^{1/2}(0) + CK \left((\text{Vol } \Omega)^{1/2} + E_0^{1/2}(0) \right) \left(e^{CMt/2} - 1 \right),$$

which yields the desired estimate. \square

4. THE GENERAL r -TH ORDER ENERGY ESTIMATES

In this section, we establish the higher order energy estimates. In view of (2.6), (2.11) and (1.5a), one has

$$\begin{aligned}
D_t \nabla^r u_a &= D_t \nabla_{a_1} \cdots \nabla_{a_r} u_a = D_t \nabla_{a_1} \cdots \nabla_{a_{r-1}} \left(\frac{\partial x^i}{\partial y^a} \frac{\partial x^{i_r}}{\partial y^{a_r}} \frac{\partial v_i}{\partial x^{i_r}} \right) \\
&= D_t \left(\frac{\partial x^i}{\partial y^a} \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \frac{\partial^r v_i}{\partial x^{i_1} \cdots \partial x^{i_r}} \right) \\
&= \frac{\partial x^i}{\partial y^a} \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \left(D_t \frac{\partial^r v_i}{\partial x^{i_1} \cdots \partial x^{i_r}} + \frac{\partial v^l}{\partial x^{i_1}} \frac{\partial^r v_i}{\partial x^l \cdots \partial x^{i_r}} + \cdots \right. \\
&\quad \left. + \frac{\partial v^l}{\partial x^{i_r}} \frac{\partial^r v_i}{\partial x^{i_1} \cdots \partial x^l} + \frac{\partial v^l}{\partial x^i} \frac{\partial^r v_l}{\partial x^{i_1} \cdots \partial x^{i_r}} \right) \\
&= \frac{\partial x^i}{\partial y^a} \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \left([D_t, \partial^r] v_i + \partial^r D_t v_i \right) + \nabla u \cdot \nabla^r u_a + \nabla_a u^c \nabla^r u_c \\
&= -\nabla^r \nabla_a p + \nabla_a u^c \nabla^r u_c - \sum_{s=1}^{r-1} \binom{r}{s+1} (\nabla^{1+s} u) \cdot \nabla^{r-s} u_a \\
&\quad + \sum_{s=0}^r \binom{r}{s} \nabla^s \mathbb{F}^{cb} \nabla^{r-s} \nabla_c \mathbb{F}_{ab},
\end{aligned}$$

where

$$\left(\nabla^s \mathbb{F}^{cb} \nabla^{r-s} \nabla_c \mathbb{F}_{ab} \right)_{a_1 \cdots a_r} = \sum_{\Sigma_r} \nabla_{a_{\sigma_1} \cdots a_{\sigma_s}}^s \mathbb{F}^{cb} \nabla_{a_{\sigma_{s+1}} \cdots a_{\sigma_r}}^{r-s} \nabla_c \mathbb{F}_{ab}.$$

Then, using $\text{div } \mathbb{F}^\top = 0$, we obtain, for $r \geq 2$,

$$D_t \nabla^r u_a + \nabla^r \nabla_a p = (\text{curl } u)_{ac} \nabla^r u^c + \text{sgn}(2-r) \sum_{s=1}^{r-2} \binom{r}{s+1} (\nabla^{1+s} u) \cdot \nabla^{r-s} u_a$$

$$+ \nabla_c \left(\mathbb{F}^{cb} \nabla^r \mathbb{F}_{ab} \right) + \sum_{s=1}^r \binom{r}{s} \nabla^s \mathbb{F}^{cb} \nabla^{r-s} \nabla_c \mathbb{F}_{ab}. \quad (4.1)$$

Similarly, by $\operatorname{div} \mathbb{F}^\top = 0$ again, we have, for $r \geq 2$,

$$\begin{aligned} D_t \nabla^r \mathbb{F}_{ab} &= D_t \left(\frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \frac{\partial^r F_{ij}}{\partial x^{i_1} \cdots \partial x^{i_r}} \right) \\ &= \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \left(D_t \frac{\partial^r F_{ij}}{\partial x^{i_1} \cdots \partial x^{i_r}} + \frac{\partial v^l}{\partial x^{i_1}} \frac{\partial^r F_{ij}}{\partial x^l \cdots \partial x^{i_r}} + \cdots \right. \\ &\quad \left. + \frac{\partial v^l}{\partial x^{i_r}} \frac{\partial^r F_{ij}}{\partial x^{i_1} \cdots \partial x^l} + \frac{\partial v^l}{\partial x^i} \frac{\partial^r F_{lj}}{\partial x^{i_1} \cdots \partial x^{i_r}} + \frac{\partial v^l}{\partial x^j} \frac{\partial^r F_{il}}{\partial x^{i_1} \cdots \partial x^{i_r}} \right) \\ &= \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \left([D_t, \partial^r] F_{ij} + \partial^r D_t F_{ij} \right) \\ &\quad + \nabla u \cdot \nabla^r \mathbb{F}_{ab} + \nabla_a u^c \nabla^r \mathbb{F}_{cb} + \nabla_b u^c \nabla^r \mathbb{F}_{ac} \\ &= \nabla_a u^c \nabla^r \mathbb{F}_{cb} + \nabla_b u^c \nabla^r \mathbb{F}_{ac} - \nabla^r u^c \nabla_c \mathbb{F}_{ab} + \nabla_c \left(g^{cd} \mathbb{F}_{db} \nabla^r u_a \right) \\ &\quad - \operatorname{sgn}(2-r) \sum_{s=1}^{r-2} \binom{r}{s+1} (\nabla^{1+s} u) \cdot \nabla^{r-s} \mathbb{F}_{ab} \\ &\quad + \sum_{s=1}^r \binom{r}{s} g^{cd} \nabla^s \mathbb{F}_{db} \nabla^{r-s} \nabla_c u_a. \end{aligned} \quad (4.2)$$

From Lemmas 2.1 and A.4, and (4.1), it follows that

$$\begin{aligned} &D_t \left(g^{bd} \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a u_b \nabla_F^{r-1} \nabla_f u_d \right) \\ &= (D_t g^{bd}) \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a u_b \nabla_F^{r-1} \nabla_f u_d + r g^{bd} (D_t \gamma^{af}) \gamma^{AF} \nabla_A^{r-1} \nabla_a u_b \nabla_F^{r-1} \nabla_f u_d \\ &\quad + 2 g^{bd} \gamma^{af} \gamma^{AF} D_t (\nabla_A^{r-1} \nabla_a u_b) \nabla_F^{r-1} \nabla_f u_d \\ &= -2 \nabla_c u_e \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a u^c \nabla_F^{r-1} \nabla_f u^e - 2 r \nabla_c u_e \gamma^{ac} \gamma^{ef} \gamma^{AF} \nabla_A^{r-1} \nabla_a u^d \nabla_F^{r-1} \nabla_f u_d \\ &\quad - 2 \nabla_b \left(\gamma^{af} \gamma^{AF} \nabla_F^{r-1} \nabla_f u^b \nabla_A^{r-1} \nabla_a p \right) \\ &\quad + 2 \nabla_b \left(\gamma^{af} \gamma^{AF} \right) \nabla_F^{r-1} \nabla_f u^b \nabla_A^{r-1} \nabla_a p + 2 \gamma^{af} \gamma^{AF} \nabla_F^{r-1} \nabla_f u^b (\operatorname{curl} u)_{bc} \nabla_A^{r-1} \nabla_a u^c \\ &\quad + 2 \operatorname{sgn}(2-r) \gamma^{af} \gamma^{AF} \nabla_F^{r-1} \nabla_f u_d \sum_{s=1}^{r-2} \binom{r}{s+1} \left((\nabla^{s+1} u) \cdot \nabla^{r-s} u^d \right)_{Aa} \\ &\quad + 2 \gamma^{af} \gamma^{AF} \nabla_F^{r-1} \nabla_f u^d \nabla_c \left(\mathbb{F}^{cb} \nabla_A^{r-1} \nabla_a \mathbb{F}_{db} \right) \end{aligned} \quad (4.3)$$

$$\begin{aligned} &+ 2 \gamma^{af} \gamma^{AF} \nabla_F^{r-1} \nabla_f u^d \sum_{s=1}^r \binom{r}{s} \left(\nabla^s \mathbb{F}^{cb} \nabla^{r-s} \nabla_c \mathbb{F}_{db} \right)_{Aa}. \end{aligned} \quad (4.4)$$

Similarly, we have

$$\begin{aligned} &D_t \left(g^{bd} g^{ce} \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a \mathbb{F}_{bc} \nabla_F^{r-1} \nabla_f \mathbb{F}_{de} \right) \\ &= D_t (g^{bd}) g^{ce} \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a \mathbb{F}_{bc} \nabla_F^{r-1} \nabla_f \mathbb{F}_{de} + g^{bd} D_t (g^{ce}) \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a \mathbb{F}_{bc} \nabla_F^{r-1} \nabla_f \mathbb{F}_{de} \\ &\quad + r D_t (\gamma^{af}) \gamma^{AF} \nabla_A^{r-1} \nabla_a \mathbb{F}_{bc} \nabla_F^{r-1} \nabla_f \mathbb{F}^{bc} + 2 \gamma^{af} \gamma^{AF} D_t (\nabla_A^{r-1} \nabla_a \mathbb{F}_{bc}) \nabla_F^{r-1} \nabla_f \mathbb{F}^{bc} \end{aligned}$$

$$\begin{aligned}
&= -2\nabla_k u^m \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a \mathbb{F}^{kc} \nabla_F^{r-1} \nabla_f \mathbb{F}_{mc} - 2\nabla_k u^m \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a \mathbb{F}^{dk} \nabla_F^{r-1} \nabla_f \mathbb{F}_{dm} \\
&\quad - 2r \nabla_d u^e \gamma^{ad} \gamma^{ef} \gamma^{AF} \nabla_A^{r-1} \nabla_a \mathbb{F}^{bc} \nabla_F^{r-1} \nabla_f \mathbb{F}_{bc} + 2\gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a \mathbb{F}_{ec} \nabla_b u^e \nabla_F^{r-1} \nabla_f \mathbb{F}^{bc} \\
&\quad + 2\gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a \mathbb{F}_{be} \nabla_c u^e \nabla_F^{r-1} \nabla_f \mathbb{F}^{bc} - 2\gamma^{af} \gamma^{AF} \nabla_F^{r-1} \nabla_f \mathbb{F}^{bc} \nabla_A^{r-1} \nabla_a u^e \nabla_e \mathbb{F}_{bc} \\
&\quad + 2\gamma^{af} \gamma^{AF} \nabla_F^{r-1} \nabla_f \mathbb{F}^{bc} g^{ed} \mathbb{F}_{dc} \nabla_e \nabla_A^{r-1} \nabla_a u_b \\
&\quad + 2\text{sgn}(2-r) \gamma^{af} \gamma^{AF} \nabla_F^{r-1} \nabla_f \mathbb{F}^{bc} \sum_{s=1}^{r-2} \binom{r}{s+1} ((\nabla^{1+s} u) \cdot \nabla^{r-s} \mathbb{F}_{bc})_{Aa} \\
&\quad + 2\gamma^{af} \gamma^{AF} \nabla_F^{r-1} \nabla_f \mathbb{F}^{bc} \sum_{s=1}^r \binom{r}{s} \left(g^{ed} \nabla^s \mathbb{F}_{dc} \nabla^{r-s} \nabla_e u_b \right)_{Aa}.
\end{aligned} \tag{4.5}$$

For (4.4) and (4.5), one has

$$\begin{aligned}
(4.4) + (4.5) &= 2\gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a u^b \nabla_e (\mathbb{F}^{ec} \nabla_F^{r-1} \nabla_f \mathbb{F}_{bc}) \\
&\quad + 2\gamma^{af} \gamma^{AF} \nabla_e \nabla_A^{r-1} \nabla_a u^b \mathbb{F}^{ec} \nabla_F^{r-1} \nabla_f \mathbb{F}_{bc} \\
&= 2\nabla_e \left(\gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a u^b \mathbb{F}^{ec} \nabla_F^{r-1} \nabla_f \mathbb{F}_{bc} \right) \\
&\quad - 2\nabla_e (\gamma^{af} \gamma^{AF}) \nabla_A^{r-1} \nabla_a u^b \mathbb{F}^{ec} \nabla_F^{r-1} \nabla_f \mathbb{F}_{bc}.
\end{aligned} \tag{4.6}$$

$$- 2\nabla_e (\gamma^{af} \gamma^{AF}) \nabla_A^{r-1} \nabla_a u^b \mathbb{F}^{ec} \nabla_F^{r-1} \nabla_f \mathbb{F}_{bc}. \tag{4.7}$$

The boundary integral stemmed from the integration of (4.6) over Ω will vanish since it involves the term $N_e \mathbb{F}^{ec}$ which is zero on the boundary. Since (3.12), especially the integral involving p , vanishes, we do not need the boundary integral in the first order energy $E_1(t)$. However, the boundary integral derived from the integral of (4.3) over Ω will be out of control for higher order energies. Thus, we have to include a boundary integral to overcome this difficulty.

Define the r -th order energy for an integer $r \geq 2$ as

$$\begin{aligned}
E_r(t) &= \int_{\Omega} g^{bd} \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a u_b \nabla_F^{r-1} \nabla_f u_d d\mu_g + \int_{\Omega} |\nabla^{r-1} \text{curl } u|^2 d\mu_g \\
&\quad + \int_{\Omega} g^{bd} g^{ce} \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a \mathbb{F}_{bc} \nabla_F^{r-1} \nabla_f \mathbb{F}_{de} d\mu_g + \int_{\Omega} |\nabla^{r-1} \text{curl } \mathbb{F}^\top|^2 d\mu_g \\
&\quad + \int_{\partial\Omega} \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a p \nabla_F^{r-1} \nabla_f p \vartheta d\mu_\gamma,
\end{aligned}$$

where $\vartheta = 1/(-\nabla_N p)$ as before. Then, we have the following theorem.

Theorem 4.1. *For the integer $r \in \{2, \dots, n+1\}$, there exists a constant $T > 0$ such that, for any smooth solution to system (2.20) for $0 \leq t \leq T$ satisfying*

$$|\mathbb{F}| \leq M_1 \quad \text{for } r = 2, \quad \text{in } [0, T] \times \Omega, \tag{4.8}$$

$$|\nabla p| \leq M, \quad |\nabla u| \leq M, \quad |\nabla \mathbb{F}| \leq M, \quad \text{in } [0, T] \times \Omega, \tag{4.9}$$

$$|\theta| + 1/\iota_0 \leq K, \quad \text{on } [0, T] \times \partial\Omega, \tag{4.10}$$

$$-\nabla_N p \geq \varepsilon > 0, \quad \text{on } [0, T] \times \partial\Omega, \tag{4.11}$$

$$|\nabla^2 p| + |\nabla_N D_t p| \leq L, \quad \text{on } [0, T] \times \partial\Omega, \tag{4.12}$$

the following estimate holds for any $t \in [0, T]$,

$$E_r(t) \leq e^{C_1 t} E_r(0) + C_2 (e^{C_1 t} - 1), \tag{4.13}$$

where the constants C_1 and C_2 depend on $K, K_1, M, M_1, L, 1/\varepsilon, \text{Vol } \Omega, E_0(0), E_1(0), \dots$, and $E_{r-1}(0)$.

Proof. The derivative of $E_r(t)$ with respect to t is

$$\frac{d}{dt}E_r(t) = \int_{\Omega} D_t \left(g^{bd} \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a u_b \nabla_F^{r-1} \nabla_f u_d \right) d\mu_g \quad (4.14)$$

$$+ \int_{\Omega} D_t \left(g^{bd} g^{ce} \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a \mathbb{F}_{bc} \nabla_F^{r-1} \nabla_f \mathbb{F}_{de} \right) d\mu_g \quad (4.15)$$

$$+ \int_{\Omega} D_t |\nabla^{r-1} \text{curl } u|^2 d\mu_g + \int_{\Omega} D_t |\nabla^{r-1} \text{curl } \mathbb{F}^\top|^2 d\mu_g \quad (4.16)$$

$$+ \int_{\Omega} g^{bd} \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a u_b \nabla_F^{r-1} \nabla_f u_d \text{tr } h d\mu_g \quad (4.17)$$

$$+ \int_{\Omega} |\nabla^{r-1} \text{curl } u|^2 \text{tr } h d\mu_g + \int_{\Omega} |\nabla^{r-1} \text{curl } \mathbb{F}^\top|^2 \text{tr } h d\mu_g \quad (4.18)$$

$$+ \int_{\Omega} g^{bd} g^{ce} \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a \mathbb{F}_{bc} \nabla_F^{r-1} \nabla_f \mathbb{F}_{de} \text{tr } h d\mu_g \quad (4.19)$$

$$+ \int_{\partial\Omega} D_t \left(\gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a p \nabla_F^{r-1} \nabla_f p \right) \vartheta d\mu_\gamma \quad (4.20)$$

$$+ \int_{\partial\Omega} \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a p \nabla_F^{r-1} \nabla_f p \left(\frac{\vartheta_t}{\vartheta} + \text{tr } h - h_{NN} \right) \vartheta d\mu_\gamma. \quad (4.21)$$

Step 1: Estimate the integrals (4.14), (4.15) and (4.20).

From the previous derivations for the integrands in (4.14) and (4.15), (4.6), (4.7) and

$$\begin{aligned} & D_t \left(\gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a p \nabla_F^{r-1} \nabla_f p \right) \\ &= -2r \nabla_c u_e \gamma^{ac} \gamma^{ef} \gamma^{AF} \nabla_A^{r-1} \nabla_a p \nabla_F^{r-1} \nabla_f p + 2\gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a p D_t \left(\nabla_F^{r-1} \nabla_f p \right), \end{aligned}$$

we have

$$(4.14) + (4.15) + (4.20)$$

$$\leq C \left(\|\nabla u\|_{L^\infty(\Omega)} + \|\nabla \mathbb{F}\|_{L^\infty(\Omega)} \right) E_r(t) + C E_r^{1/2}(t) \sum_{s=1}^{r-2} \|\nabla^{s+1} u\|_{L^4(\Omega)} \left(\|\nabla^{r-s} u\|_{L^4(\Omega)} + \|\nabla^{r-s} \mathbb{F}\|_{L^4(\Omega)} \right) \quad (4.22)$$

$$+ C E_r^{1/2}(t) \sum_{s=2}^{r-1} \|\nabla^s \mathbb{F}\|_{L^4(\Omega)} \left(\|\nabla^{r-s+1} u\|_{L^4(\Omega)} + \|\nabla^{r-s+1} \mathbb{F}\|_{L^4(\Omega)} \right) \quad (4.23)$$

$$+ 2 \int_{\partial\Omega} \gamma^{af} \gamma^{AF} \nabla_A^r \left(D_t \nabla_F^r p - \frac{1}{\vartheta} N_b \nabla_F^r u^b \right) \vartheta d\mu_\gamma \quad (4.24)$$

$$+ 2 \int_{\Omega} \nabla_b \left(\gamma^{af} \gamma^{AF} \right) \nabla_F^{r-1} \nabla_f u^b \nabla_A^{r-1} \nabla_a p d\mu_g \quad (4.25)$$

$$+ \int_{\partial\Omega} N_e \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a u^b \mathbb{F}^{ec} \nabla_F^{r-1} \nabla_f \mathbb{F}_{bc} d\mu_\gamma \quad (4.26)$$

$$- \int_{\Omega} \nabla_e \left(\gamma^{af} \gamma^{AF} \right) \nabla_A^{r-1} \nabla_a u^b \mathbb{F}^{ec} \nabla_F^{r-1} \nabla_f \mathbb{F}_{bc} d\mu_g. \quad (4.27)$$

Since $N \cdot \mathbb{F}^\top = 0$ on $\partial\Omega$, (4.26) vanishes. From Lemma A.12, we see that, for $\iota_1 \geq 1/K_1$,

$$\|\mathbb{F}\|_{L^\infty(\Omega)} \leq C \sum_{0 \leq s \leq 2} K_1^{n/2-s} \|\nabla^s \mathbb{F}\|_{L^2(\Omega)} \leq C(K_1) \sum_{s=0}^2 E_s^{1/2}(t). \quad (4.28)$$

Using the Hölder inequality and the assumption (4.9), we obtain for any integers $r \geq 3$,

$$|(4.27)| \leq CK \|\mathbb{F}\|_{L^\infty(\Omega)} E_r(t) \leq C(K, K_1) \left(\sum_{s=0}^2 E_s^{1/2}(t) \right) E_r(t). \quad (4.29)$$

For $r = 2$, by (4.8), one has

$$|(4.27)| \leq CK \|\mathbb{F}\|_{L^\infty(\Omega)} E_r(t) \leq C(K, M_1) E_r(t). \quad (4.30)$$

Step 1.1: Estimate (4.25).

From Hölder's inequality, we have

$$|(4.25)| \leq CK E_r^{1/2}(t) \|\nabla^r p\|_{L^2(\Omega)}. \quad (4.31)$$

It follows from (2.20a) and (2.14) that

$$\Delta p = -\nabla_a u^b \nabla_b u^a + g^{cb} \nabla_a \mathbb{F}_{cd} \nabla_b \mathbb{F}^{ad}. \quad (4.32)$$

Then, for $r \geq 2$,

$$\begin{aligned} \nabla^{r-2} \Delta p &= - \sum_{s=0}^{r-2} \binom{r-2}{s} \nabla^s \nabla_a u^b \nabla^{r-2-s} \nabla_b u^a \\ &\quad + \sum_{s=0}^{r-2} \binom{r-2}{s} g^{cb} \nabla^s \nabla_a \mathbb{F}_{cd} \nabla^{r-2-s} \nabla_b \mathbb{F}^{ad}. \end{aligned}$$

In view of (4.28), one has, for $s \geq 0$,

$$\|\nabla^s \mathbb{F}\|_{L^\infty(\Omega)} \leq C \sum_{\ell=0}^2 K_1^{n/2-\ell} \|\nabla^{\ell+s} \mathbb{F}\|_{L^2(\Omega)} \leq C(K_1) \sum_{\ell=0}^2 E_{s+\ell}^{1/2}(t), \quad (4.33)$$

and

$$\|\nabla^s u\|_{L^\infty(\Omega)} \leq C(K_1) \sum_{\ell=0}^2 E_{s+\ell}^{1/2}(t). \quad (4.34)$$

From the Hölder inequality, (4.33) and (4.34), we have, for $r \in \{3, 4\}$,

$$\begin{aligned} &\|\nabla^{r-2} \Delta p\|_{L^2(\Omega)} \\ &\leq C \sum_{s=0}^{r-2} \left\| \nabla^s \nabla_a u^b \nabla^{r-2-s} \nabla_b u^a \right\|_{L^2(\Omega)} \\ &\quad + C \sum_{s=0}^{r-2} \left\| g^{cb} \nabla^s \nabla_a \mathbb{F}_{cd} \nabla^{r-2-s} \nabla_b \mathbb{F}^{ad} \right\|_{L^2(\Omega)} \\ &\leq C \|\nabla u\|_{L^\infty(\Omega)} \|\nabla^{r-1} u\|_{L^2(\Omega)} + C \|\nabla \mathbb{F}\|_{L^\infty(\Omega)} \|\nabla^{r-1} \mathbb{F}\|_{L^2(\Omega)} \end{aligned}$$

$$\begin{aligned}
& + (r-3)C \left(\|\nabla^2 u\|_{L^\infty(\Omega)} \|\nabla^2 u\|_{L^2(\Omega)} + \|\nabla^2 \mathbb{F}\|_{L^\infty(\Omega)} \|\nabla^2 \mathbb{F}\|_{L^2(\Omega)} \right) \\
& \leq C(K_1) \sum_{\ell=1}^{r-1} E_\ell(t) + C(K_1) E_2^{1/2}(t) E_r^{1/2}(t).
\end{aligned} \tag{4.35}$$

For the case $r = 2$, we have the following estimate from the assumption (4.9) and the Hölder inequality:

$$\|\Delta p\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)} \|\nabla u\|_{L^\infty(\Omega)} + C \|\nabla \mathbb{F}\|_{L^2(\Omega)} \|\nabla \mathbb{F}\|_{L^\infty(\Omega)} \leq C M E_1^{1/2}(t), \tag{4.36}$$

which is a lower order energy term. Hence, from (A.12), (4.35) and (4.36), we have for any real number $\delta_r > 0$,

$$\begin{aligned}
\|\nabla^r p\|_{L^2(\Omega)} & \leq \delta_r \|\Pi \nabla^r p\|_{L^2(\partial\Omega)} + C(1/\delta_r, K, \text{Vol } \Omega) \sum_{s \leq r-2} \|\nabla^s \Delta p\|_{L^2(\Omega)} \\
& \leq \delta_r \|\Pi \nabla^r p\|_{L^2(\partial\Omega)} + C(1/\delta_r, K, K_1, M, \text{Vol } \Omega) \sum_{\ell=1}^{r-1} E_\ell(t) \\
& \quad + (r-2)C(1/\delta_r, K, K_1, M, \text{Vol } \Omega) E_2^{1/2}(t) E_r^{1/2}(t).
\end{aligned} \tag{4.37}$$

Next, we estimate the boundary terms. Because $p = 0$ on the boundary $\partial\Omega$, from (A.13), we obtain for $r \geq 1$,

$$\begin{aligned}
\|\Pi \nabla^r p\|_{L^2(\partial\Omega)} & \leq C(K, K_1) \left(\|\theta\|_{L^\infty(\partial\Omega)} + (r-2) \sum_{k \leq r-3} \left\| \overline{\nabla}^k \theta \right\|_{L^2(\partial\Omega)} \right) \\
& \quad \times \sum_{k \leq r-1} \left\| \nabla^k p \right\|_{L^2(\partial\Omega)}.
\end{aligned} \tag{4.38}$$

Due to (A.7), we have $\Pi \nabla^2 p = \theta \nabla_N p$. From (4.11), (4.10), (A.18), (4.9) and (4.37), we obtain

$$\|\theta\|_{L^2(\partial\Omega)} = \left\| \frac{\Pi \nabla^2 p}{\nabla_N p} \right\|_{L^2(\partial\Omega)} \leq \frac{1}{\varepsilon} \|\Pi \nabla^2 p\|_{L^2(\partial\Omega)}, \tag{4.39}$$

and

$$\begin{aligned}
\|\Pi \nabla^2 p\|_{L^2(\partial\Omega)} & \leq \|\theta\|_{L^\infty(\partial\Omega)} \|\nabla p\|_{L^2(\partial\Omega)} \leq C(K, \text{Vol } \Omega) \left(\|\nabla^2 p\|_{L^2(\Omega)} + \|\nabla p\|_{L^2(\Omega)} \right) \\
& \leq C(K, \text{Vol } \Omega) \delta_2 \|\Pi \nabla^2 p\|_{L^2(\partial\Omega)} + C(K, \text{Vol } \Omega) (\text{Vol } \Omega)^{1/2} M \\
& \quad + C(1/\delta_2, K, K_1, M, \text{Vol } \Omega) E_1(t),
\end{aligned} \tag{4.40}$$

where the first term on the right hand side of (4.40) can be absorbed by the left hand side if we take δ_2 small such that $C(K, \text{Vol } \Omega) \delta_2 \leq 1/2$. Then,

$$\|\Pi \nabla^2 p\|_{L^2(\partial\Omega)} + \|\nabla^2 p\|_{L^2(\Omega)} \leq C(K, K_1, M, \text{Vol } \Omega) (1 + E_1(t)), \tag{4.41}$$

$$\|\theta\|_{L^2(\partial\Omega)} \leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon) (1 + E_1(t)). \tag{4.42}$$

From Theorem 3.1, there exists a constant $T > 0$ such that $E_1(t)$ can be controlled by the initial energy $E_1(0)$ for $t \in [0, T]$, e.g., $E_1(t) \leq 2E_1(0)$. Then, from (4.38),

(4.42), (4.9) and (4.41), we get

$$\begin{aligned} \|\Pi \nabla^3 p\|_{L^2(\partial\Omega)} &\leq C(K, K_1) \left(K + \|\theta\|_{L^2(\partial\Omega)} \right) \sum_{k \leq 2} \|\nabla^k p\|_{L^2(\partial\Omega)} \\ &\leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) \|\nabla^3 p\|_{L^2(\Omega)} \\ &\quad + C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)). \end{aligned}$$

It follows from (4.37) that

$$\begin{aligned} \|\nabla^3 p\|_{L^2(\Omega)} &\leq \delta_3 C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) \|\nabla^3 p\|_{L^2(\Omega)} \\ &\quad + \delta_3 C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) \\ &\quad + C(1/\delta_3, K, K_1, M, \text{Vol } \Omega)(E_1(t) + E_2(t)) \\ &\quad + C(1/\delta_3, K, K_1, M, \text{Vol } \Omega) E_2^{1/2}(t) E_3^{1/2}(t), \end{aligned}$$

which, if we take $\delta_3 > 0$ so small that $\delta_3 C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) \leq 1/2$, implies

$$\begin{aligned} \|\nabla^3 p\|_{L^2(\Omega)} &\leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) + C(K, K_1, M, \text{Vol } \Omega) \sum_{\ell=1}^2 E_\ell(t) \\ &\quad + C(K, K_1, M, \text{Vol } \Omega) E_2^{1/2}(t) E_3^{1/2}(t), \end{aligned} \tag{4.43}$$

and thus

$$\|\Pi \nabla^3 p\|_{L^2(\partial\Omega)} \leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) \left(1 + \sum_{\ell=1}^2 E_\ell(t) + E_2^{1/2}(t) E_3^{1/2}(t) \right).$$

Because

$$\begin{aligned} \bar{\nabla}_b \nabla_N p &= \gamma_b^d \nabla_d (N^a \nabla_a p) = (\delta_b^d - N_b N^d) ((\nabla_d N^a) \nabla_a p + N^a \nabla_d \nabla_a p) \\ &= \theta_b^a \nabla_a p + N^a \nabla_b \nabla_a p - N_b N^d (\theta_d^a \nabla_a p + N^a \nabla_d \nabla_a p), \end{aligned}$$

we have from (A.18) that

$$\begin{aligned} \|\bar{\nabla} \nabla_N p\|_{L^2(\partial\Omega)} &\leq C \|\theta\|_{L^\infty(\partial\Omega)} \|\nabla p\|_{L^2(\partial\Omega)} + C \|\nabla^2 p\|_{L^2(\partial\Omega)} \\ &\leq C(K, \text{Vol } \Omega) \left(\|\nabla^3 p\|_{L^2(\Omega)} + \|\nabla^2 p\|_{L^2(\Omega)} + \|\nabla p\|_{L^2(\Omega)} \right) \\ &\leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) + C(K, K_1, M, \text{Vol } \Omega) \sum_{\ell=1}^2 E_\ell(t) \\ &\quad + C(K, K_1, M, \text{Vol } \Omega) E_2^{1/2}(t) E_3^{1/2}(t). \end{aligned}$$

Then, by (A.8), we have

$$(\bar{\nabla} \theta) \nabla_N p = \Pi \nabla^3 p - 3\theta \bar{\otimes} \bar{\nabla} \nabla_N p$$

and

$$\begin{aligned} \|\bar{\nabla} \theta\|_{L^2(\partial\Omega)} &\leq \frac{1}{\varepsilon} \left(\|\Pi \nabla^3 p\|_{L^2(\partial\Omega)} + C \|\theta\|_{L^\infty(\partial\Omega)} \|\bar{\nabla} \nabla_N p\|_{L^2(\partial\Omega)} \right) \\ &\leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) \left(1 + \sum_{\ell=1}^2 E_\ell(t) + E_2^{1/2}(t) E_3^{1/2}(t) \right). \end{aligned}$$

From (4.38) and (A.18),

$$\|\Pi \nabla^4 p\|_{L^2(\partial\Omega)} \leq C(K, K_1) \left(K + \|\theta\|_{L^2(\partial\Omega)} + \|\bar{\nabla}\theta\|_{L^2(\partial\Omega)} \right) \sum_{k \leq 4} \|\nabla^k p\|_{L^2(\Omega)}. \quad (4.44)$$

Thus, by (4.37) we can absorb the highest order term $\|\nabla^4 p\|_{L^2(\Omega)}$ by the left hand side for $\delta_4 > 0$ small enough which is independent of the highest order energy $E_4(t)$, and

$$\begin{aligned} & \|\nabla^4 p\|_{L^2(\Omega)} + \|\Pi \nabla^4 p\|_{L^2(\partial\Omega)} \\ & \leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) \left(1 + \sum_{\ell=1}^3 E_\ell(t) + E_2^{1/2}(t) E_4^{1/2}(t) \right). \end{aligned}$$

Hence, from (4.41), (4.43) and (4.44), we have for $r \geq 2$,

$$\|\nabla^r p\|_{L^2(\Omega)} \leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) \left(1 + \sum_{\ell=1}^{r-1} E_\ell(t) + (r-2) E_2^{1/2}(t) E_r^{1/2}(t) \right),$$

which, from (4.31), yields

$$|(4.25)| \leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) E_r^{1/2}(t) \left(1 + \sum_{\ell=1}^{r-1} E_\ell(t) + (r-2) E_2^{1/2}(t) E_r^{1/2}(t) \right).$$

Step 1.2: Estimate (4.24).

The boundary condition $p = 0$ on $\partial\Omega$ implies $\gamma_b^a \nabla_a p = 0$ on $\partial\Omega$. Then we have, from (A.3) and $\vartheta = -1/\nabla_N p$,

$$-\vartheta^{-1} N_b = \nabla_N p N_b = N^a \nabla_a p N_b = \delta_b^a \nabla_a p - \gamma_b^a \nabla_a p = \nabla_b p. \quad (4.45)$$

From the Hölder inequality and (4.45), we get

$$\begin{aligned} |(4.24)| & \leq C \|\vartheta\|_{L^\infty(\partial\Omega)}^{1/2} E_r^{1/2}(t) \left\| \Pi \left(D_t (\nabla^r p) - \vartheta^{-1} N_b \nabla^r u^b \right) \right\|_{L^2(\partial\Omega)} \\ & = C \|\vartheta\|_{L^\infty(\partial\Omega)}^{1/2} E_r^{1/2}(t) \left\| \Pi (D_t (\nabla^r p) + \nabla^r u \cdot \nabla p) \right\|_{L^2(\partial\Omega)}. \end{aligned} \quad (4.46)$$

It follows from (2.17) that

$$\begin{aligned} D_t \nabla^r p + \nabla^r u \cdot \nabla p & = [D_t, \nabla^r] p + \nabla^r D_t p + \nabla^r u \cdot \nabla p \\ & = \text{sgn}(2-r) \sum_{s=1}^{r-2} \binom{r}{s+1} (\nabla^{s+1} u) \cdot \nabla^{r-s} p + \nabla^r D_t p. \end{aligned} \quad (4.47)$$

Now, we consider the last term in (4.47). From (A.13) and (A.18), we have, for $2 \leq r \leq 4$,

$$\begin{aligned} & \|\Pi \nabla^r D_t p\|_{L^2(\partial\Omega)} \\ & \leq C(K, K_1, \text{Vol } \Omega) \left(\|\theta\|_{L^\infty(\partial\Omega)} + (r-2) \sum_{k \leq r-3} \|\bar{\nabla}^k \theta\|_{L^2(\partial\Omega)} \right) \sum_{k \leq r} \|\nabla^k D_t p\|_{L^2(\Omega)}. \end{aligned} \quad (4.48)$$

It follows from (A.12) that

$$\|\nabla^r D_t p\|_{L^2(\Omega)} \leq \delta \|\Pi \nabla^r D_t p\|_{L^2(\partial\Omega)} + C(1/\delta, K, \text{Vol } \Omega) \sum_{s \leq r-2} \|\nabla^s \Delta D_t p\|_{L^2(\Omega)}. \quad (4.49)$$

From (2.16), (4.32), Lemma 2.1, (3.1), (3.2) and (2.20), it follows that

$$\begin{aligned} \Delta D_t p &= 2h^{ab} \nabla_a \nabla_b p + (\Delta u^e) \nabla_e p - D_t(g^{bd} g^{ac} \nabla_a u_d \nabla_b u_c) + D_t(g^{cb} g^{ae} g^{df} \nabla_a \mathbb{F}_{cd} \nabla_b \mathbb{F}_{ef}) \\ &= 2h^{ab} \nabla_a \nabla_b p + (\Delta u^e) \nabla_e p - 2D_t(g^{bd}) \nabla_a u_d \nabla_b u^a - 2g^{bd} D_t(\nabla_a u_d) \nabla_b u^a \\ &\quad + 2D_t(g^{cb}) \nabla_a \mathbb{F}_{cd} \nabla_b \mathbb{F}^{ad} + g^{cb} g^{ae} D_t(g^{df}) \nabla_a \mathbb{F}_{cd} \nabla_b \mathbb{F}_{ef} + 2g^{cb} D_t(\nabla_a \mathbb{F}_{cd}) \nabla_b \mathbb{F}^{ad} \\ &= 2g^{ac} \nabla_c u^b \nabla_a \nabla_b p + (\Delta u^e) \nabla_e p + 2\nabla_e u^b \nabla_b u^a \nabla_a u^e - 2g^{ce} \nabla_e u^b \nabla_a \mathbb{F}_{cd} \nabla_b \mathbb{F}^{ad} \\ &\quad - 2\nabla_d u_f \nabla_a \mathbb{F}^{bd} \nabla_b \mathbb{F}^{af} + 2g^{ce} \nabla_c \mathbb{F}^{ad} \nabla_a \mathbb{F}_{eb} \nabla_d u^b - 2g^{bd} \nabla_b u^a \nabla_a \nabla_c \mathbb{F}_{de} \mathbb{F}^{ce} \\ &\quad + 2g^{ce} \nabla_b \mathbb{F}^{ad} \mathbb{F}_{ed} \nabla_a \nabla_c u^b. \end{aligned}$$

From (4.33), (4.37) and Lemma A.12, it implies that, for $s \leq 2$,

$$\begin{aligned} &\|\nabla^s \Delta D_t p\|_{L^2(\Omega)} \\ &\leq C \|\nabla u\|_{L^\infty(\Omega)} \|\nabla^{s+2} p\|_{L^2(\Omega)} + s(s-1)C \|\nabla^3 u\|_{L^2(\Omega)} \|\nabla^2 p\|_{L^\infty(\Omega)} \\ &\quad + sC \|\nabla^2 u\|_{L^4(\Omega)} \|\nabla^{s+1} p\|_{L^4(\Omega)} + C \|\nabla^{s+2} u\|_{L^2(\Omega)} \|\nabla p\|_{L^\infty(\Omega)} \\ &\quad + C \left(\|\nabla u\|_{L^\infty(\Omega)} \|\nabla u\|_{L^\infty(\Omega)} + \|\nabla \mathbb{F}\|_{L^\infty(\Omega)} \|\nabla \mathbb{F}\|_{L^\infty(\Omega)} \right) \|\nabla^{s+1} u\|_{L^2(\Omega)} \\ &\quad + s(s-1)C \|\nabla u\|_{L^\infty(\Omega)} \|\nabla^2 u\|_{L^4(\Omega)} \|\nabla^2 u\|_{L^4(\Omega)} \\ &\quad + C \|\nabla u\|_{L^\infty(\Omega)} \|\nabla \mathbb{F}\|_{L^\infty(\Omega)} \|\nabla^{s+1} \mathbb{F}\|_{L^2(\Omega)} \\ &\quad + sC \|\nabla^2 u\|_{L^4(\Omega)} \|\nabla^2 \mathbb{F}\|_{L^4(\Omega)} \left((s-1) \|\nabla \mathbb{F}\|_{L^\infty(\Omega)} + \|\mathbb{F}\|_{L^\infty(\Omega)} \right) \\ &\quad + s(s-1)C \|\nabla u\|_{L^\infty(\Omega)} \|\nabla^2 \mathbb{F}\|_{L^4(\Omega)} \|\nabla^2 \mathbb{F}\|_{L^4(\Omega)} \\ &\quad + C \|\nabla u\|_{L^\infty(\Omega)} \|\mathbb{F}\|_{L^\infty(\Omega)} \|\nabla^{s+2} \mathbb{F}\|_{L^2(\Omega)} \\ &\quad + sC \|\nabla^3 u\|_{L^2(\Omega)} \|\mathbb{F}\|_{L^\infty(\Omega)} \left((s-1) \|\nabla^2 \mathbb{F}\|_{L^\infty(\Omega)} + \|\nabla \mathbb{F}\|_{L^\infty(\Omega)} \right) \\ &\quad + s(s-1)C \|\nabla^3 \mathbb{F}\|_{L^2(\Omega)} \|\mathbb{F}\|_{L^\infty(\Omega)} \|\nabla^2 u\|_{L^\infty(\Omega)} \\ &\quad + s(s-1)C \|\nabla \mathbb{F}\|_{L^\infty(\Omega)} \|\nabla^2 \mathbb{F}\|_{L^4(\Omega)} \|\nabla^2 u\|_{L^4(\Omega)} \\ &\quad + s(s-1)C \|\nabla \mathbb{F}\|_{L^\infty(\Omega)} \|\mathbb{F}\|_{L^\infty(\Omega)} \|\nabla^4 u\|_{L^2(\Omega)} \\ &\quad + s(s-1)C \|\nabla^2 \mathbb{F}\|_{L^\infty(\Omega)} \|\mathbb{F}\|_{L^\infty(\Omega)} \|\nabla^3 u\|_{L^2(\Omega)}. \end{aligned}$$

In view of Lemma A.11 and (4.34), the following holds

$$\|\nabla^{s+1} u\|_{L^4(\Omega)} \leq C \|\nabla^s u\|_{L^\infty(\Omega)}^{1/2} \left(\sum_{\ell=0}^2 \|\nabla^{s+\ell} u\|_{L^2(\Omega)} K_1^{2-\ell} \right)^{1/2} \leq C(K_1) \sum_{\ell=0}^2 E_{s+\ell}^{1/2}(t).$$

We can estimate all the terms with $L^4(\Omega)$ norms in the same way in view of (4.33), (4.34), the similar estimate of p and the assumptions. Hence, we obtain the bound

which is linear with respect to the highest-order derivative or the highest-order energy $E_r^{1/2}(t)$, i.e.,

$$\|\nabla^s \Delta D_t p\|_{L^2(\Omega)} \leq C(K, K_1, M, M_1, L, 1/\varepsilon, \text{Vol } \Omega, E_0(0)) \left(1 + \sum_{\ell=0}^{r-1} E_\ell(t)\right) (1 + E_r^{1/2}(t)). \quad (4.50)$$

Therefore, by (4.48), (4.49), (4.50) and for some small δ independent of $E_r(t)$, we obtain, by the induction argument for r ,

$$\begin{aligned} \|\Pi \nabla^r D_t p\|_{L^2(\partial\Omega)} &\leq C(K, K_1, M, M_1, L, 1/\varepsilon, \text{Vol } \Omega, E_0(0)) \\ &\quad \times \left(1 + \sum_{\ell=0}^{r-1} E_\ell(t)\right) (1 + E_r^{1/2}(t)). \end{aligned} \quad (4.51)$$

For the estimate of (4.47), it only remains to estimate

$$\|\Pi((\nabla^{s+1} u) \cdot \nabla^{r-s} p)\|_{L^2(\partial\Omega)} \quad \text{for } 1 \leq s \leq r-2.$$

For the cases $r = 3, 4$ and $s = r-2$, we have, from (4.12) and Lemma A.14, that

$$\begin{aligned} &\|\Pi((\nabla^{r-1} u) \cdot \nabla^2 p)\|_{L^2(\partial\Omega)} \\ &\leq \|\nabla^{r-1} u\|_{L^2(\partial\Omega)} \|\nabla^2 p\|_{L^\infty(\partial\Omega)} \leq CL \|\nabla^2 u\|_{L^{2(n-1)/(n-2)}(\partial\Omega)} \\ &\leq C(K, \text{Vol } \Omega) L \left(\|\nabla^r u\|_{L^2(\Omega)} + \|\nabla^{r-1} u\|_{L^2(\Omega)}\right) \\ &\leq C(K, L, \text{Vol } \Omega) \left(E_{r-1}^{1/2}(t) + E_r^{1/2}(t)\right). \end{aligned}$$

For the cases $n = 3$, $r = 4$ and $s = 1$, from (A.6), Lemma A.14 and (4.37), we have

$$\begin{aligned} &\|\Pi((\nabla^2 u) \cdot \nabla^3 p)\|_{L^2(\partial\Omega)} \\ &= \|\Pi \nabla^2 u \cdot \Pi \nabla^3 p + \Pi(\nabla^2 u \cdot N) \tilde{\otimes} \Pi(N \cdot \nabla^3 p)\|_{L^2(\partial\Omega)} \\ &\leq C \|\Pi \nabla^2 u\|_{L^4(\partial\Omega)} \|\Pi \nabla^3 p\|_{L^4(\partial\Omega)} + C \|\Pi(N^a \nabla^2 u_a)\|_{L^4(\partial\Omega)} \|\Pi(\nabla_N \nabla^2 p)\|_{L^4(\partial\Omega)} \\ &\leq C \|\nabla^2 u\|_{L^4(\partial\Omega)} \|\nabla^3 p\|_{L^4(\partial\Omega)} \\ &\leq C(K, \text{Vol } \Omega) \left(\|\nabla^3 u\|_{L^2(\Omega)} + \|\nabla^2 u\|_{L^2(\Omega)}\right) \left(\|\nabla^4 p\|_{L^2(\Omega)} + \|\nabla^3 p\|_{L^2(\Omega)}\right) \\ &\leq C(K, K_1, \text{Vol } \Omega) (E_3^{1/2}(t) + E_2^{1/2}(t)) \left(\sum_{s=0}^3 E_s(t) + \left(\sum_{\ell=0}^2 E_\ell^{1/2}(t)\right) E_4^{1/2}(t)\right) \\ &\leq C(K, K_1, \text{Vol } \Omega) \sum_{s=0}^3 E_s(t) \sum_{\ell=0}^4 E_\ell^{1/2}(t). \end{aligned}$$

Thus, we get

$$|(4.24)| \leq C(K, K_1, M, M_1, L, 1/\varepsilon, \text{Vol } \Omega, E_0(0)) \left(1 + \sum_{s=0}^{r-1} E_s(t)\right) (1 + E_r(t)).$$

From Lemma A.11, it follows that

$$|(4.22) + (4.23)| \leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon) \left(1 + \sum_{s=0}^{r-1} E_s(t)\right) E_r(t).$$

Therefore, we have shown that

$$\begin{aligned} & |(4.14) + (4.15) + (4.20)| \\ & \leq C(K, K_1, M, M_1, L, 1/\varepsilon, \text{Vol } \Omega, E_0(0)) \left(1 + \sum_{s=0}^{r-1} E_s(t)\right) (1 + E_r(t)). \end{aligned}$$

Step 2: Estimate (4.16)-(4.19) and (4.21).

From Lemma 2.1, (4.1) and (4.2), we have

$$\begin{aligned} & D_t \left(|\nabla^{r-1} \text{curl } u|^2 + |\nabla^{r-1} \text{curl } \mathbb{F}^\top|^2 \right) \\ &= D_t \left(g^{ac} g^{bd} g^{AF} \nabla_A^{r-1} (\text{curl } u)_{ab} \nabla_F^{r-1} (\text{curl } u)_{cd} \right) \\ & \quad + D_t \left(g^{ac} g^{bd} g^{ef} g^{AF} \nabla_A^{r-1} (\text{curl } \mathbb{F}^\top)_{abe} \nabla_F^{r-1} (\text{curl } \mathbb{F}^\top)_{cdf} \right) \\ &= (r+1) D_t (g^{ac}) g^{bd} g^{AF} \nabla_A^{r-1} (\text{curl } u)_{ab} \nabla_F^{r-1} (\text{curl } u)_{cd} \\ & \quad + 4g^{ac} g^{bd} g^{AF} D_t (\nabla_A^{r-1} \nabla_a u_b) \nabla_F^{r-1} (\text{curl } u)_{cd} \\ & \quad + (r+1) D_t (g^{ac}) g^{bd} g^{ef} g^{AF} \nabla_A^{r-1} (\text{curl } \mathbb{F}^\top)_{abe} \nabla_F^{r-1} (\text{curl } \mathbb{F}^\top)_{cdf} \\ & \quad + g^{ac} g^{bd} D_t (g^{ef}) g^{AF} \nabla_A^{r-1} (\text{curl } \mathbb{F}^\top)_{abe} \nabla_F^{r-1} (\text{curl } \mathbb{F}^\top)_{cdf} \\ & \quad + 4g^{ac} g^{bd} g^{ef} g^{AF} D_t (\nabla_A^{r-1} \nabla_a \mathbb{F}_{be}) \nabla_F^{r-1} (\text{curl } \mathbb{F}^\top)_{cdf} \\ &= -2(r+1) g^{ae} \nabla_e u^c g^{bd} g^{AF} \nabla_A^{r-1} (\text{curl } u)_{ab} \nabla_F^{r-1} (\text{curl } u)_{cd} \\ & \quad - 2(r+1) g^{ae} \nabla_e u^c g^{bd} g^{ef} g^{AF} \nabla_A^{r-1} (\text{curl } \mathbb{F}^\top)_{abe} \nabla_F^{r-1} (\text{curl } \mathbb{F}^\top)_{cdf} \\ & \quad + 2g^{ac} g^{bd} g^{es} \nabla_s u^f g^{AF} \nabla_A^{r-1} (\text{curl } \mathbb{F}^\top)_{abe} \nabla_F^{r-1} (\text{curl } \mathbb{F}^\top)_{cdf} \\ & \quad + 4g^{ac} g^{bd} g^{AF} \nabla_F^{r-1} (\text{curl } u)_{cd} (\text{curl } u)_{be} \nabla_{Aa}^r u^e \\ & \quad + 4\text{sgn}(2-r) g^{ac} g^{AF} \nabla_F^{r-1} (\text{curl } u)_{cd} \sum_{s=1}^{r-2} \binom{r}{s+1} \left((\nabla^{1+s} u) \cdot \nabla^{r-s} u^d \right)_{Aa} \\ & \quad + 4\text{sgn}(2-r) g^{ac} g^{AF} \nabla_F^{r-1} (\text{curl } \mathbb{F}^\top)_{cdf} \sum_{s=1}^{r-2} \binom{r}{s+1} \left((\nabla^{1+s} u) \cdot \nabla^{r-s} \mathbb{F}^{df} \right)_{Aa} \\ & \quad + 4g^{ac} g^{bd} g^{ef} g^{AF} \nabla_F^{r-1} (\text{curl } \mathbb{F}^\top)_{cdf} \nabla_{Aa}^r \mathbb{F}_{se} \nabla_b u^s \\ & \quad - 4g^{ac} g^{bd} g^{ef} g^{AF} \nabla_F^{r-1} (\text{curl } \mathbb{F}^\top)_{cdf} \nabla_{Aa}^r u^s \nabla_s \mathbb{F}_{be} \\ & \quad + 4g^{ac} g^{bd} g^{ef} g^{AF} \nabla_F^{r-1} (\text{curl } \mathbb{F}^\top)_{cdf} \nabla_e u^s \nabla_{Aa}^r \mathbb{F}_{bs} \\ & \quad + 4\nabla_f \left(g^{ac} g^{bd} g^{AF} \mathbb{F}^{fe} \nabla_F^{r-1} (\text{curl } u)_{cd} \nabla_{Aa}^r \mathbb{F}_{be} \right) \\ & \quad + 4g^{ac} g^{bd} g^{AF} \nabla_F^{r-1} (\text{curl } u)_{cd} \sum_{s=1}^r \binom{r}{s} \left(\nabla^s \mathbb{F}^{ef} \nabla^{r-s} \nabla_e \mathbb{F}^{bf} \right)_{Aa} \end{aligned}$$

$$+ 4g^{ac}g^{AF}\nabla_F^{r-1}(\operatorname{curl}\mathbb{F}^\top)_{cdf}\sum_{s=1}^r\binom{r}{s}\left(\nabla^s\mathbb{F}^{bf}\nabla^{r-s}\nabla_b u^d\right)_{Aa}.$$

Since $N \cdot \mathbb{F}^\top = 0$ on $\partial\Omega$, by the Hölder inequality and the Gauss formula, we have

$$(4.16) \leq C(K, K_1, M, \operatorname{Vol}\Omega, 1/\varepsilon) \left(1 + \sum_{s=0}^{r-1} E_s(t)\right) E_r(t). \quad (4.52)$$

From (A.9) and (2.16), we get

$$\begin{aligned} D_t(\nabla_N p) &= D_t(N^a \nabla_a p) = (D_t N^a) \nabla_a p + N^a D_t \nabla_a p \\ &= (-2h_d^a N^d + h_{NN} N^a) \nabla_a p + N^a \nabla_a D_t p \\ &= -2h_d^a N^d \nabla_a p + h_{NN} \nabla_N p + \nabla_N D_t p, \end{aligned}$$

which implies

$$\frac{\vartheta_t}{\vartheta} = -\frac{D_t \nabla_N p}{\nabla_N p} = \frac{2h_d^a N^d \nabla_a p}{\nabla_N p} - h_{NN} + \frac{\nabla_N D_t p}{\nabla_N p}. \quad (4.53)$$

Hence, (4.21) can be controlled by $C(K, M, L, 1/\varepsilon)E_r(t)$. The remaining integrals (4.17), (4.18) and (4.19) vanish due to the fact $\operatorname{tr} h = 0$.

Therefore, we have

$$\frac{d}{dt} E_r(t) \leq C(K, K_1, M, M_1, L, 1/\varepsilon, \operatorname{Vol}\Omega, E_0(0)) \left(1 + \sum_{s=0}^{r-1} E_s(t)\right) (1 + E_r(t)), \quad (4.54)$$

which implies the desired result (4.13) by the Gronwall inequality and the induction argument for $r \in \{2, \dots, n+1\}$. \square

5. JUSTIFICATION OF A PRIORI ASSUMPTIONS

In the derivation of the higher order energy estimates in Section 4, some *a priori* assumptions are made. In this section we shall justify these *a priori* assumptions.

Denote

$$\begin{aligned} \mathcal{K}(t) &= \max \left(\|\theta(t, \cdot)\|_{L^\infty(\partial\Omega)}, 1/\iota_0(t) \right), \\ \mathcal{E}(t) &= \|1/(\nabla_N p(t, \cdot))\|_{L^\infty(\partial\Omega)}, \quad \varepsilon(t) = \frac{1}{\mathcal{E}(t)}. \end{aligned} \quad (5.1)$$

As in Definition A.3, let $0 < \varepsilon_1 < 2$ be a fixed number, take $\iota_1 = \iota_1(\varepsilon_1)$ to be the largest number such that $|\mathcal{N}(\bar{x}_1) - \mathcal{N}(\bar{x}_2)| \leq \varepsilon_1$ whenever $|\bar{x}_1 - \bar{x}_2| \leq \iota_1$ for $\bar{x}_1, \bar{x}_2 \in \partial\mathcal{D}_t$.

Lemma 5.1. *Let $K_1 \geq 1/\iota_1$. Then there are continuous functions G_j , $j = 1, 2, 3, 4$, such that*

$$\|\nabla u\|_{L^\infty(\Omega)} + \|\nabla \mathbb{F}\|_{L^\infty(\Omega)} + \|\mathbb{F}\|_{L^\infty(\Omega)} \leq G_1(K_1, E_0, \dots, E_{n+1}), \quad (5.2)$$

$$\|\nabla p\|_{L^\infty(\Omega)} + \|\nabla^2 p\|_{L^\infty(\partial\Omega)} \leq G_2(K_1, \mathcal{E}, E_0, \dots, E_{n+1}, \operatorname{Vol}\Omega), \quad (5.3)$$

$$\|\theta\|_{L^\infty(\partial\Omega)} \leq G_3(K_1, \mathcal{E}, E_0, \dots, E_{n+1}, \operatorname{Vol}\Omega), \quad (5.4)$$

$$\|\nabla D_t p\|_{L^\infty(\partial\Omega)} \leq G_4(K_1, \mathcal{E}, E_0, \dots, E_{n+1}, \operatorname{Vol}\Omega). \quad (5.5)$$

Proof. The estimate (5.2) follows from (4.34), (4.33) and (4.28). By Lemmas A.12 and A.10, we obtain

$$\|\nabla p\|_{L^\infty(\Omega)} \leq C(K_1) \sum_{\ell=0}^2 \left\| \nabla^{\ell+1} p \right\|_{L^2(\Omega)}, \quad (5.6)$$

and

$$\|\nabla^2 p\|_{L^\infty(\partial\Omega)} \leq C(K_1) \sum_{\ell=0}^{n+1} \left\| \nabla^\ell p \right\|_{L^2(\partial\Omega)}. \quad (5.7)$$

Thus, the estimate (5.3) follows from (5.6), (5.7), Lemmas A.13–A.14, (4.36), (4.41) and (4.43). Since $|\nabla^2 p| \geq |\Pi \nabla^2 p| = |\nabla_N p| |\theta| \geq \mathcal{E}^{-1} |\theta|$ in view of (A.7), the estimate (5.4) follows from (5.3). The estimate (5.5) follows from Lemma A.10, (4.49), (4.50) and (4.51). \square

Lemma 5.2. *Let $K_1 \geq 1/\iota_1$. Then we have*

$$\left| \frac{d}{dt} E_r \right| \leq C_r(K_1, \mathcal{E}, E_0, \dots, E_{n+1}, \text{Vol } \Omega) \sum_{s=0}^r E_s, \quad (5.8)$$

and

$$\left| \frac{d}{dt} \mathcal{E} \right| \leq C_r(K_1, \mathcal{E}, E_0, \dots, E_{n+1}, \text{Vol } \Omega). \quad (5.9)$$

Proof. The first estimate (5.8) follows immediately from Lemma 5.1 and the proof of Theorems 3.1 and 4.1. The second estimate follows from

$$\left| \frac{d}{dt} \left\| \frac{1}{-\nabla_N p(t, \cdot)} \right\|_{L^\infty(\partial\Omega)} \right| \leq C \left\| \frac{1}{-\nabla_N p(t, \cdot)} \right\|_{L^\infty(\partial\Omega)}^2 \|\nabla_N D_t p(t, \cdot)\|_{L^\infty(\partial\Omega)}$$

and (5.5). \square

As a consequence of Lemma 5.2, we have the following result:

Lemma 5.3. *There exists a continuous function $\mathcal{T} > 0$ depending on $K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0)$ and $\text{Vol } \Omega$ such that for*

$$0 \leq t \leq \mathcal{T}(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \text{Vol } \Omega), \quad (5.10)$$

one has

$$E_s(t) \leq 2E_s(0), \quad 0 \leq s \leq n+1, \quad \mathcal{E}(t) \leq 2\mathcal{E}(0). \quad (5.11)$$

Furthermore,

$$\frac{1}{2} g_{ab}(0, y) Y^a Y^b \leq g_{ab}(t, y) Y^a Y^b \leq 2g_{ab}(0, y) Y^a Y^b, \quad (5.12)$$

and

$$|\mathcal{N}(x(t, \bar{y})) - \mathcal{N}(x(0, \bar{y}))| \leq \frac{\varepsilon_1}{16}, \quad \bar{y} \in \partial\Omega, \quad (5.13)$$

$$|x(t, y) - x(0, y)| \leq \frac{\iota_1}{16}, \quad y \in \Omega, \quad (5.14)$$

$$\left| \frac{\partial x(t, \bar{y})}{\partial y} - \frac{\partial x(0, \bar{y})}{\partial y} \right| \leq \frac{\varepsilon_1}{16}, \quad \bar{y} \in \partial\Omega. \quad (5.15)$$

Proof. First when $\mathcal{T}(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \text{Vol } \Omega) > 0$ is sufficiently small, Lemma 5.2 yields the estimate (5.11). Then, from (5.11) and Lemma 5.1,

$$\begin{aligned} \|\nabla u\|_{L^\infty(\Omega)} + \|\nabla \mathbb{F}\|_{L^\infty(\Omega)} + \|\mathbb{F}\|_{L^\infty(\Omega)} + \|\nabla p\|_{L^\infty(\Omega)} \\ \leq C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0)), \end{aligned} \quad (5.16)$$

$$\|\nabla^2 p\|_{L^\infty(\partial\Omega)} + \|\theta\|_{L^\infty(\partial\Omega)} \leq C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \text{Vol } \Omega), \quad (5.17)$$

and

$$\|\nabla D_t p\|_{L^\infty(\partial\Omega)} \leq C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \text{Vol } \Omega). \quad (5.18)$$

From (3.1) and (3.2), we have

$$|D_t \nabla u| \leq |\nabla^2 p| + |\nabla u|^2 + |\nabla \mathbb{F}|^2 + |\mathbb{F}| |\nabla^2 \mathbb{F}|, \quad |D_t \nabla \mathbb{F}| \leq |\nabla \mathbb{F}| |\nabla u| + |\mathbb{F}| |\nabla^2 u|.$$

With the help of (A.15), (A.18), Lemma 5.1 and (5.11), we obtain

$$\|\nabla u\|_{L^\infty(\partial\Omega)} + \|\nabla \mathbb{F}\|_{L^\infty(\partial\Omega)} \leq C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \text{Vol } \Omega).$$

It follows, from (5.16), (5.17), Lemmas A.10 and A.14, (4.34) and (4.33), that

$$\begin{aligned} & \|D_t \nabla u\|_{L^\infty(\partial\Omega)} + \|D_t \nabla \mathbb{F}\|_{L^\infty(\partial\Omega)} \\ & \leq \|\nabla^2 p\|_{L^\infty(\partial\Omega)} + \left(\|\nabla u\|_{L^\infty(\partial\Omega)} + \|\nabla \mathbb{F}\|_{L^\infty(\partial\Omega)} \right)^2 \\ & \quad + \|\mathbb{F}\|_{L^\infty(\Omega)} \left(\|\nabla^2 u\|_{L^\infty(\partial\Omega)} + \|\nabla^2 \mathbb{F}\|_{L^\infty(\partial\Omega)} \right) \\ & \leq C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \text{Vol } \Omega) \\ & \quad \times \left(1 + \|\nabla u\|_{L^\infty(\partial\Omega)} + \|\nabla \mathbb{F}\|_{L^\infty(\partial\Omega)} \right), \end{aligned}$$

which implies, by Gronwall's inequality, for $t \geq 0$

$$\begin{aligned} & \|\nabla u(t, \cdot)\|_{L^\infty(\partial\Omega)} + \|\nabla \mathbb{F}(t, \cdot)\|_{L^\infty(\partial\Omega)} \\ & \leq e^{C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \text{Vol } \Omega)t} \left(\|\nabla u(0, \cdot)\|_{L^\infty(\partial\Omega)} + \|\nabla \mathbb{F}(0, \cdot)\|_{L^\infty(\partial\Omega)} \right) \\ & \quad + e^{C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \text{Vol } \Omega)t} - 1. \end{aligned}$$

It follows, for $0 \leq t \leq T$, that

$$\begin{aligned} & \|\nabla u(t, \cdot)\|_{L^\infty(\partial\Omega)} + \|\nabla \mathbb{F}(t, \cdot)\|_{L^\infty(\partial\Omega)} \\ & \leq 2 \left(\|\nabla u(0, \cdot)\|_{L^\infty(\partial\Omega)} + \|\nabla \mathbb{F}(0, \cdot)\|_{L^\infty(\partial\Omega)} \right), \end{aligned} \quad (5.19)$$

if we take T small enough, which also guarantees the *a priori* assumption of (2.22).

From (2.16), (A.17), (4.49), (4.50) and (4.51), we obtain

$$\begin{aligned} \|D_t \nabla p\|_{L^\infty(\Omega)} &= \|\nabla D_t p\|_{L^\infty(\Omega)} \leq C(K_1) \sum_{\ell=0}^2 \left\| \nabla^{\ell+1} D_t p \right\|_{L^2(\Omega)} \\ &\leq C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \text{Vol } \Omega), \end{aligned}$$

which yields, for sufficiently small $t > 0$,

$$\|\nabla p(t, \cdot)\|_{L^\infty(\Omega)} \leq 2 \|\nabla p(0, \cdot)\|_{L^\infty(\Omega)}. \quad (5.20)$$

In view of (2.20) and (5.16), we get

$$\begin{aligned} \|D_t v\|_{L^\infty(\mathcal{D}_t)} &\leq \|\partial p\|_{L^\infty(\mathcal{D}_t)} + \|F\|_{L^\infty(\mathcal{D}_t)} \|\partial F\|_{L^\infty(\mathcal{D}_t)} \\ &\leq \|\nabla p\|_{L^\infty(\Omega)} + \|\mathbb{F}\|_{L^\infty(\Omega)} \|\nabla \mathbb{F}\|_{L^\infty(\Omega)} \\ &\leq C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0)), \end{aligned}$$

which implies

$$\|v(t, \cdot)\|_{L^\infty(\mathcal{D}_t)} \leq 2 \|v(0, \cdot)\|_{L^\infty(\Omega)}. \quad (5.21)$$

The relation (5.12) follows from the same argument because $D_t g_{ab} = \nabla_a u_b + \nabla_b u_a$ and by (5.16)

$$\begin{aligned} &\left| g_{ab}(T, y) Y^a Y^b - g_{ab}(0, y) Y^a Y^b \right| \\ &\leq \int_0^T |D_t g_{ab}(s, y)| ds Y^a Y^b \leq 2 \int_0^T \|\nabla_a u_b(s)\|_{L^\infty(\Omega)} ds Y^a Y^b \leq \frac{1}{2} g_{ab}(0, y) Y^a Y^b, \end{aligned}$$

as long as T is sufficiently small.

Recalling the fact

$$D_t n_a = h_{NN} n_a,$$

and

$$D_t x(t, y) = v(t, x(t, y)), \quad D_t \frac{\partial x}{\partial y} = \frac{\partial v(t, x(t, y))}{\partial y} = \frac{\partial v(t, x)}{\partial x} \frac{\partial x}{\partial y},$$

one obtains (5.13)-(5.15) from (5.21) and (5.19). \square

As a consequence of (5.13), (5.14) and the triangle inequality, we have the following result:

Lemma 5.4. *Let \mathcal{T} be as in Lemma 5.3. There exists some $\iota_1 > 0$ such that, if*

$$|\mathcal{N}(x(0, y_1)) - \mathcal{N}(x(0, y_2))| \leq \frac{\varepsilon_1}{2}$$

when $|x(0, y_1) - x(0, y_2)| \leq 2\iota_1$, then

$$|\mathcal{N}(x(t, y_1)) - \mathcal{N}(x(t, y_2))| \leq \varepsilon_1$$

when $|x(t, y_1) - x(t, y_2)| \leq 2\iota_1$.

Lemmas 5.3 and 5.4 yield immediately the main Theorem 1.1.

APPENDIX A. PRELIMINARIES AND SOME ESTIMATES

For the convenience of the readers and completeness of preliminary results, we record some definitions and estimates directly from Christodoulou-Lindblad [3] in this appendix.

Let N^a denote the unit normal to $\partial\Omega$, $g_{ab} N^a N^b = 1$, $g_{ab} N^a T^b = 0$ if $T \in T(\partial\Omega)$, and let $N_a = g_{ab} N^b$ denote the unit conormal, $g^{ab} N_a N_b = 1$. The induced metric γ on the tangent space to the boundary $T(\partial\Omega)$ extended to be 0 on the orthogonal complement in $T(\Omega)$ is then given by

$$\gamma_{ab} = g_{ab} - N_a N_b, \quad \gamma^{ab} = g^{ab} - N^a N^b. \quad (\text{A.1})$$

The orthogonal projection of an (r, s) tensor S to the boundary is given by

$$(\Pi S)_{b_1 \dots b_s}^{a_1 \dots a_r} = \gamma_{c_1}^{a_1} \dots \gamma_{c_r}^{a_r} \gamma_{b_1}^{d_1} \dots \gamma_{b_s}^{d_s} S_{d_1 \dots d_s}^{c_1 \dots c_r}, \quad (\text{A.2})$$

where

$$\gamma_a^c = \delta_a^c - N_a N^c. \quad (\text{A.3})$$

Covariant differentiation on the boundary $\bar{\nabla}$ is given by

$$\bar{\nabla} S = \Pi \nabla S. \quad (\text{A.4})$$

The second fundamental form of the boundary is given by

$$\theta_{ab} = (\Pi \nabla N)_{ab} = \gamma_a^c \nabla_c N_b. \quad (\text{A.5})$$

Let us now recall some properties of the projection. Since $g^{ab} = \gamma^{ab} + N^a N^b$, we have

$$\Pi(S \cdot R) = \Pi(S) \cdot \Pi(R) + \Pi(S \cdot N) \tilde{\otimes} \Pi(N \cdot R), \quad (\text{A.6})$$

where $S \tilde{\otimes} R$ denotes some partial symmetrization of the tensor product $S \otimes R$, i.e., a sum over some subset of the permutations of the indices divided by the number of permutations in that subset. Similarly, we let $S \tilde{\cdot} R$ denote a partial symmetrization of the dot product $S \cdot R$. Now we recall some identities:

$$\Pi \nabla^2 q = \bar{\nabla}^2 q + \theta \nabla_N q, \quad (\text{A.7})$$

$$\Pi \nabla^3 q = \bar{\nabla}^3 q - 2\theta \tilde{\otimes} (\theta \cdot \bar{\nabla} q) + (\bar{\nabla} \theta) \nabla_N q + 3\theta \tilde{\otimes} \bar{\nabla} \nabla_N q. \quad (\text{A.8})$$

Definition A.1. Let $\mathcal{N}(\bar{x})$ be the outward unit normal to $\partial \mathcal{D}_t$ at $\bar{x} \in \partial \mathcal{D}_t$. Let $\text{dist}(x_1, x_2) = |x_1 - x_2|$ denote the Euclidean distance in \mathbb{R}^n , and for $\bar{x}_1, \bar{x}_2 \in \partial \mathcal{D}_t$, let $\text{dist}_{\partial \mathcal{D}_t}(\bar{x}_1, \bar{x}_2)$ denote the geodesic distance on the boundary.

Definition A.2. Let $\text{dist}(x, \partial \mathcal{D}_t)$ be the Euclidean distance from x to the boundary. Let ι_0 be the injectivity radius of the normal exponential map of $\partial \mathcal{D}_t$, i.e., the largest number such that the map

$$\begin{aligned} \partial \mathcal{D}_t \times (-\iota_0, \iota_0) &\rightarrow \{x \in \mathbb{R}^n : \text{dist}(x, \partial \mathcal{D}_t) < \iota\} \\ \text{given by } (\bar{x}, \iota) &\rightarrow x = \bar{x} + \iota \mathcal{N}(\bar{x}) \end{aligned}$$

is an injection.

Definition A.3. Let $0 < \varepsilon_1 < 2$ be a fixed number, and let $\iota_1 = \iota_1(\varepsilon_1)$ the largest number such that

$$|\mathcal{N}(\bar{x}_1) - \mathcal{N}(\bar{x}_2)| \leq \varepsilon_1 \quad \text{whenever } |\bar{x}_1 - \bar{x}_2| \leq \iota_1, \bar{x}_1, \bar{x}_2 \in \partial \mathcal{D}_t.$$

Lemma A.4 ([3, Lemma 3.9]). *Let N be the unit normal to $\partial \Omega$, and let $h_{ab} = \frac{1}{2} D_t g_{ab}$. On $[0, T] \times \partial \Omega$, we have*

$$D_t N_a = h_{NN} N_a, \quad D_t N^c = -2h_d^c N^d + h_{NN} N^c, \quad \text{and } D_t \gamma^{ab} = -2\gamma^{ac} h_{cd} \gamma^{db}, \quad (\text{A.9})$$

where $h_{NN} = h_{ab} N^a N^b$. The volume element on $\partial \Omega$ satisfies

$$D_t d\mu_\gamma = (\text{tr } h - h_{NN}) d\mu_\gamma = (\text{tr } \theta u \cdot N + \gamma^{ab} \bar{\nabla}_a \bar{u}_b) d\mu_\gamma, \quad (\text{A.10})$$

where \bar{u}_b denotes the tangential component of u_b to the boundary $\partial \Omega$.

Lemma A.5 (cf. [3, Lemma 5.5]). *Let $w_a = w_{Aa} = \nabla_A^r f_a$, $\nabla_A^r = \nabla_{a_1} \cdots \nabla_{a_r}$, f be a $(0,1)$ tensor, and $[\nabla_a, \nabla_b] = 0$. Let $\operatorname{div} w = \nabla_a w^a = \nabla^r \operatorname{div} f$, and let $(\operatorname{curl} w)_{ab} = \nabla_a w_b - \nabla_b w_a = \nabla^r (\operatorname{curl} f)_{ab}$. Then,*

$$|\nabla w|^2 \leq C(g^{ab} \gamma^{cd} \gamma^{AB} \nabla_c w_{Aa} \nabla_d w_{Bb} + |\operatorname{div} w|^2 + |\operatorname{curl} w|^2). \quad (\text{A.11})$$

Lemma A.6 ([3, Proposition 5.8]). *Let ι_0 and ι_1 be as in Definitions A.2 and A.3, and suppose that $|\theta| + 1/\iota_0 \leq K$ and $1/\iota_1 \leq K_1$. Then with $\tilde{K} = \min(K, K_1)$ we have, for any $r \geq 2$ and $\delta > 0$,*

$$\begin{aligned} & \|\nabla^r q\|_{L^2(\partial\Omega)} + \|\nabla^r q\|_{L^2(\Omega)} \\ & \leq C \|\Pi \nabla^r q\|_{L^2(\partial\Omega)} + C(\tilde{K}, \operatorname{Vol} \Omega) \sum_{s \leq r-1} \|\nabla^s \Delta q\|_{L^2(\Omega)}, \\ & \|\nabla^{r-1} q\|_{L^2(\partial\Omega)} + \|\nabla^r q\|_{L^2(\Omega)} \\ & \leq \delta \|\Pi \nabla^r q\|_{L^2(\partial\Omega)} + C(1/\delta, K, \operatorname{Vol} \Omega) \sum_{s \leq r-2} \|\nabla^s \Delta q\|_{L^2(\Omega)}. \end{aligned} \quad (\text{A.12})$$

Lemma A.7 (cf. [3, Proposition 5.9]). *Assume that $0 \leq r \leq 4$. Suppose that $|\theta| \leq K$ and $\iota_1 \geq 1/K_1$, where ι_1 is as in Definition 3.5 of [3]. If $q = 0$ on $\partial\Omega$, then for $m = 0, 1$,*

$$\begin{aligned} \|\Pi \nabla^r q\|_{L^2(\partial\Omega)} & \leq C(K, K_1) \left(\|\theta\|_{L^\infty(\partial\Omega)} + \sum_{k \leq r-2-m} \|\bar{\nabla}^k \theta\|_{L^2(\partial\Omega)} \right) \\ & \quad \times \sum_{k \leq r-2+m} \|\nabla^k q\|_{L^2(\partial\Omega)}. \end{aligned} \quad (\text{A.13})$$

If, in addition, $|\nabla_N q| \geq \varepsilon > 0$ and $|\nabla_N q| \geq 2\varepsilon \|\nabla_N q\|_{L^\infty(\partial\Omega)}$, then

$$\|\bar{\nabla}^{r-2} \theta\|_{L^2(\partial\Omega)} \leq C \left(K, K_1, \frac{1}{\varepsilon} \right) \left(\|\theta\|_{L^\infty(\partial\Omega)} + \sum_{k \leq r-3} \|\bar{\nabla}^k \theta\|_{L^2(\partial\Omega)} \right) \sum_{k \leq r-1} \|\nabla^k q\|_{L^2(\partial\Omega)}.$$

Lemma A.8 (cf. [3, Proposition 5.10]). *Assume that $0 \leq r \leq 4$ and that $|\theta| + 1/\iota_0 \leq K$. If $q = 0$ on $\partial\Omega$, then*

$$\begin{aligned} \|\nabla^{r-1} q\|_{L^2(\partial\Omega)} & \leq C \left(\|\bar{\nabla}^{r-3} \theta\|_{L^2(\partial\Omega)} \|\nabla_N q\|_{L^\infty(\partial\Omega)} + \|\nabla^{r-2} \Delta q\|_{L^2(\Omega)} \right) \\ & \quad + C \left(K, \operatorname{Vol} \Omega, \|\theta\|_{L^2(\partial\Omega)} \right) \left(\|\nabla_N q\|_{L^\infty(\partial\Omega)} + \sum_{s=0}^{r-3} \|\nabla^s \Delta q\|_{L^2(\Omega)} \right). \end{aligned}$$

Lemma A.9 ([3, Lemma A.1]). *Let $2 \leq p \leq s \leq q \leq \infty$ and $\frac{m}{s} = \frac{k}{p} + \frac{m-k}{q}$. If α is a $(0, r)$ tensor, then with $a = k/m$ and a constant C that only depends on m and n , such that*

$$\|\bar{\nabla}^k \alpha\|_{L^s(\partial\Omega)} \leq C \|\alpha\|_{L^q(\partial\Omega)}^{1-a} \|\bar{\nabla}^m \alpha\|_{L^p(\partial\Omega)}^a.$$

Lemma A.10 ([3, Lemma A.2]). *Suppose that for $\iota_1 \geq 1/K_1$*

$$|\mathcal{N}(\bar{x}_1) - \mathcal{N}(\bar{x}_2)| \leq \varepsilon_1, \quad \text{whenever } |\bar{x}_1 - \bar{x}_2| \leq \iota_1, \quad \bar{x}_1, \bar{x}_2 \in \partial\mathcal{D}_t,$$

and

$$C_0^{-1} \gamma_{ab}^0(y) Z^a Z^b \leq \gamma_{ab}(t, y) Z^a Z^b \leq C_0 \gamma_{ab}^0(y) Z^a Z^b, \quad \text{if } Z \in T(\Omega),$$

where $\gamma_{ab}^0(y) = \gamma_{ab}(0, y)$. Then if α is a $(0, r)$ tensor,

$$\|\alpha\|_{L^{(n-1)p/(n-1-kp)}(\partial\Omega)} \leq C(K_1) \sum_{\ell=0}^k \left\| \nabla^\ell \alpha \right\|_{L^p(\partial\Omega)}, \quad 1 \leq p < \frac{n-1}{k}, \quad (\text{A.14})$$

$$\|\alpha\|_{L^\infty(\partial\Omega)} \leq \delta \left\| \nabla^k \alpha \right\|_{L^p(\partial\Omega)} + C_\delta(K_1) \sum_{\ell=0}^{k-1} \left\| \nabla^\ell \alpha \right\|_{L^p(\partial\Omega)}, \quad k > \frac{n-1}{p}, \quad (\text{A.15})$$

for any $\delta > 0$.

Lemma A.11 ([3, Lemma A.3]). *With notation as in Lemmas A.9 and A.10, we have*

$$\sum_{j=0}^k \left\| \nabla^j \alpha \right\|_{L^s(\Omega)} \leq C \|\alpha\|_{L^q(\Omega)}^{1-a} \left(\sum_{i=0}^m \left\| \nabla^i \alpha \right\|_{L^p(\Omega)} K_1^{m-i} \right)^a.$$

Lemma A.12 ([3, Lemma A.4]). *Suppose that $\iota_1 \geq 1/K_1$ and α is a $(0, r)$ tensor. Then*

$$\|\alpha\|_{L^{np/(n-kp)}(\Omega)} \leq C \sum_{\ell=0}^k K_1^{k-\ell} \left\| \nabla^\ell \alpha \right\|_{L^p(\Omega)}, \quad 1 \leq p < \frac{n}{k}, \quad (\text{A.16})$$

$$\|\alpha\|_{L^\infty(\Omega)} \leq C \sum_{\ell=0}^k K_1^{n/p-\ell} \left\| \nabla^\ell \alpha \right\|_{L^p(\Omega)}, \quad k > \frac{n}{p}. \quad (\text{A.17})$$

Lemma A.13 ([3, Lemma A.5]). *Suppose that $q = 0$ on $\partial\Omega$. Then*

$$\|q\|_{L^2(\Omega)} \leq C(\text{Vol } \Omega)^{1/n} \|\nabla q\|_{L^2(\Omega)}, \quad \|\nabla q\|_{L^2(\Omega)} \leq C(\text{Vol } \Omega)^{1/2n} \|\Delta q\|_{L^2(\Omega)}.$$

Lemma A.14 ([3, Lemma A.7]). *Let α be a $(0, r)$ tensor. Assume that*

$$\text{Vol } \Omega \leq V \text{ and } \|\theta\|_{L^\infty(\partial\Omega)} + 1/\iota_0 \leq K,$$

then there is a $C = C(K, V, r, n)$ such that

$$\|\alpha\|_{L^{(n-1)p/(n-p)}(\partial\Omega)} \leq C \|\nabla \alpha\|_{L^p(\Omega)} + C \|\alpha\|_{L^p(\Omega)}, \quad 1 \leq p < n, \quad (\text{A.18})$$

$$\|\nabla^2 \alpha\|_{L^2(\Omega)} \leq C \left(\|\Pi \nabla^2 \alpha\|_{L^{2(n-1)/n}(\partial\Omega)} + \|\Delta \alpha\|_{L^2(\Omega)} + \|\nabla \alpha\|_{L^2(\Omega)} \right). \quad (\text{A.19})$$

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